Time-Varying Linear Systems: Identification and Transmission through Unidentified Channels

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Outline

1. Introduction: Linear Time-Varying Systems
2. Identification under Side Constraints
3. Signal Transmission over Unidentified Channels
4. Example: Message Transmission with unknown Channel Support

Time-Varying Linear Systems

Time-varying Linear SISO Systems

▷ Single-Input Single-Output Systems (SISO):
Linear time-varying SISO channels are described by operators of $\mathrm{H}: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ the form

$$(\mathrm{H}f)(t) = \iint_{\mathbb{R}\times\mathbb{R}} \eta_{\mathrm{H}}(\tau,\nu)\cdot f(t-\tau)\,\mathrm{e}^{\mathrm{i}2\pi\nu t}\,\mathrm{d}\nu\,\mathrm{d}\tau = \iint_{\mathbb{R}\times\mathbb{R}} \eta_{\mathrm{H}}(\tau,\nu)\,(\mathrm{M}_\nu \mathrm{T}_\tau f)(t)\,\mathrm{d}\nu\,\mathrm{d}\tau$$

with

- Spreading function: $\eta_{\mathrm{H}} \mathbb{R}\times\mathbb{R} \to \mathbb{C}$
- Translation (time-shift) operator: $(\mathrm{T}_\tau f)(t) = f(t-\tau)$
- Modulation (frequency shift): $\mathrm{M}_\nu(f)(t) = f(t)\,\mathrm{e}^{\mathrm{i}2\pi\nu t}$

Time-varying Linear MIMO Systems

▷ Multiple-Input Multiple-Output Systems (MIMO):
Channels with N-inputs and M-outputs are characterized by operators $\mathbf{H} : \left(L^2(\mathbb{R})\right)^N \to \left(L^2(\mathbb{R})\right)^M$ of the form

$$\mathbf{H}\begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix} = \begin{pmatrix} \mathrm{H}_{1,1} & \cdots & \mathrm{H}_{1,N} \\ \vdots & & \vdots \\ \mathrm{H}_{M,1} & \cdots & \mathrm{H}_{M,N} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix} = \begin{pmatrix} \sum_{n=1}^{N} \mathrm{H}_{1,n} f_n \\ \vdots \\ \sum_{n=1}^{N} \mathrm{H}_{M,n} f_n \end{pmatrix} .$$

Each subchannel $H_{m,n}$ is a TVL SISO system.

Finite-dimensional TVL Channels

- The identification problem of TVL systems $\mathbf{H} : (L^2(\mathbb{R}))^N \to (L^2(\mathbb{R}))^M$ can be reduced to a finite-dimensional problem $\mathbf{H} : (\mathbb{C}^L)^N \to (\mathbb{C}^L)^M$

$$\mathbf{H}\begin{pmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_N \end{pmatrix} = \begin{pmatrix} \mathbf{H}_{1,1} & \cdots & \mathbf{H}_{1,N} \\ \vdots & & \vdots \\ \mathbf{H}_{M,1} & \cdots & \mathbf{H}_{M,N} \end{pmatrix} \begin{pmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_N \end{pmatrix} = \begin{pmatrix} \sum_{n=1}^N \mathbf{H}_{1,n}\mathbf{x}_n \\ \vdots \\ \sum_{n=1}^N \mathbf{H}_{M,n}\mathbf{x}_n \end{pmatrix}.$$

wherein each sub-system $\mathbf{H}_{n,m} : \mathbb{C}^L \to \mathbb{C}^L$ has the form

$$\mathbf{H}_{m,n}\mathbf{x} = \sum_{\ell=0}^{L-1}\sum_{k=0}^{L-1} \eta_{m,n}(k,\ell)\,\mathbf{M}^{\ell}\mathbf{T}^{k}\mathbf{x} = \mathbf{G}(\mathbf{x})\boldsymbol{\eta}$$

with spreading coefficients $\{\eta_{m,n}(k,\ell)\}_{k,\ell=0}^{L-1}$, and with the translation operator $\mathbf{T} : \mathbb{C}^L \to \mathbb{C}^L$ and the modulation operator $\mathbf{M} : \mathbb{C}^L \to \mathbb{C}^L$ given by

$$\mathbf{T} : \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{L-1} \end{pmatrix} \mapsto \begin{pmatrix} x_{L-1} \\ x_0 \\ \vdots \\ x_{L-2} \end{pmatrix} \quad \text{and} \quad \mathbf{M} : \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{L-1} \end{pmatrix} \mapsto \begin{pmatrix} x_0 \\ x_1 \mathrm{e}^{\mathrm{i}\frac{2\pi}{L}\cdot 1} \\ \vdots \\ x_{L-1}\mathrm{e}^{\mathrm{i}\frac{2\pi}{L}\cdot(L-1)} \end{pmatrix}$$

Operator Paley–Wiener spaces

- Set of all linear operators $\mathbb{C}^L \to \mathbb{C}^L$ is spanned by time-frequency shifts $\mathbf{M}^\ell \mathbf{T}^k$

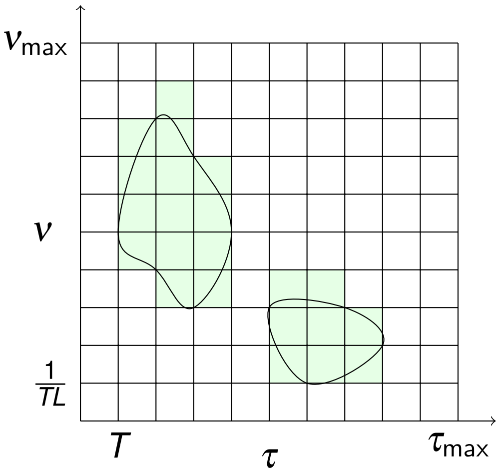
$$\mathscr{L}(\mathbb{C}^L) = \left\{ \mathbf{H} = \sum_{k=0}^{L-1} \sum_{\ell=0}^{L-1} \eta(k,\ell) \mathbf{M}^\ell \mathbf{T}^k \ : \ \eta(k,\ell) \in \mathbb{C} \text{ for all } (k,\ell) \in \mathbb{Z}_L \times \mathbb{Z}_L \right\}$$

- SISO Operator Paley–Wiener space: For $\Lambda \subset \mathbb{Z}_L \times \mathbb{Z}_L$

$$OPW(\Lambda) = \operatorname{span}\left\{ \mathbf{M}^\ell \mathbf{T}^k \ : \ (k,\ell) \in \Lambda \right\}$$

- MIMO Operator Paley–Wiener space: For $\boldsymbol{\Lambda} = \{\Lambda_{m,n}\}_{m,n=1}^{M,N}$ with $\Lambda_{m,n} \in \mathbb{Z}_L \times \mathbb{Z}_L$

$$OPW(\boldsymbol{\Lambda}) = \{ \mathbf{H} : \mathbf{H}_{m,n} \in OPW(\Lambda_{m,n}) \}$$



Identification under linear side constraints

Identification – SISO

Definition (Identifiable)

The space $OPW(\Lambda)$ is identifiable if and only if there exists an identifier $\mathbf{c} \in \mathbb{C}^L$ such that for each $\mathbf{H} \in OPW(\Lambda)$ the equation

$$\mathbf{y} = \mathbf{Hc} = \sum_{(k,\ell)\in\Lambda} \eta(k,\ell)\,\mathbf{M}^{\ell}\mathbf{T}^{k}\,\mathbf{c} = \mathbf{G}(\mathbf{c})\boldsymbol{\eta}$$

is uniquely solvable for $\boldsymbol{\eta} \in \mathbb{C}^{\Lambda}$.

Remark

$\mathbf{G}(\mathbf{c})$ is Gabor matrix of size $L \times L^2$

$$\mathbf{G}(\mathbf{c}) = \left[\mathbf{M}^0\mathbf{T}^0\,\mathbf{c}\,, \mathbf{M}^0\mathbf{T}^1\,\mathbf{c}\,, \cdots\,, \mathbf{M}^{L-1}\mathbf{T}^{L-1}\,\mathbf{c}\,\right]$$

Theorem (Identificaltion of SISO Channels)

The space $OPW(\Lambda)$ is identifiable if and only if $|\Lambda| \leq L$.

Identification – MIMO

Definition (Identifiable MIMO)

The space $OPW(\boldsymbol{\Lambda})$ is identifiable if and only if there exist vectors $\mathbf{c} = (\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_N) \in (\mathbb{C}^L)^N$ such that for each $\mathbf{H} \in OPW(\boldsymbol{\Lambda})$ the map $\mathbf{H} \mapsto \mathbf{y} = \mathbf{H}\mathbf{c}$ is injective.

Theorem (Identification of MIMO Channels)

The space $OPW(\Lambda)$ is identifiable if and only if

$$\sum_{n=1}^{N} |\Lambda_{m,n}| \leq L \qquad \textit{for every } m = 1, \dots, M\,.$$

Assumptions

- Subchannels $\mathbf{H}_{m,n}$ and their time-frequency components $\eta_{m,n}(k,\ell)$ are independent
- Information about one channel does not help to identify another.

Linear Constrains Between Spreading Coefficients

- Channels **H** in $OPW(\Lambda) \subset \mathscr{L}(\mathbb{C}^L)$:

$$\mathbf{y} = \mathbf{H}\mathbf{c} = \sum_{(k,\ell)\in\Lambda} \eta(k,\ell)\,\mathbf{M}^{\ell}\mathbf{T}^{k}\,\mathbf{c} = \mathbf{G}(\mathbf{c})\boldsymbol{\eta}$$

- $OPW(\Lambda)$ is identifiable if $|\Lambda| \leq L$.
- Assume linear relations between the spreading coefficients are known

$$\sum_{k,\ell} \alpha_{k,\ell}\,\eta(k,\ell) = \beta \qquad \text{for some} \quad \alpha_{k,\ell}, \beta \in \mathbb{C}$$

Intuition/Question

- Let $\mathbf{A}\boldsymbol{\eta} = \mathbf{b}$ be a given set of $M \geq 1$ linear independent side constraints.
- Let $OPW_{\mathbf{A},\mathbf{b}}(\Lambda)$ be the set of all $\mathbf{H} \in OPW(\Lambda)$ which satisfy these side constraints.
- $OPW_{\mathbf{A},\mathbf{b}}(\Lambda)$ is identifiable if and only if $|\Lambda| \leq L + M$?

Linear Relations between Time-Frequency Components

▷ In the SISO case, linear relations between the spreading coefficients of the channel are expressed by

$$\mathbf{b} = \mathbf{A}\boldsymbol{\eta}$$

▷ Including the equation for channel identification yields

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} \mathbf{G}(\mathbf{c}) \\ \mathbf{A} \end{bmatrix} \boldsymbol{\eta}$$

$\rightarrow$ More equations for the same number of unknowns should help to identify the channel.

Theorem (One linear side constraint)

Let $\mathbf{H} \in OPW(\Lambda)$ with $\Lambda \subset \mathbb{Z}_L \times \mathbb{Z}_L$ with $|\Lambda| = L+1$ and let $\mathbf{a} \in \mathbb{C}^{L+1}$, $\mathbf{a} \neq 0$. There there exists a $\mathbf{c} \in \mathbb{C}^L$ so that

$$\begin{bmatrix} \mathbf{G}(\mathbf{c})|_\Lambda \\ \mathbf{a}^* \end{bmatrix}$$

is invertible. Thus the channel coefficients $\boldsymbol{\eta}$ are identifiable.
Moreover, the set of all such identifiers $\mathbf{c}$ constitute a dense open subset of $\mathbb{C}^L$.

More Side Constrains

Lemma (No general solution for more than one side constraint)

Let $\mathbf{H} \in OPW(\Lambda)$ *with* $\Lambda \subset \mathbb{Z}_L \times \mathbb{Z}_L$ *with* $|\Lambda| > L+1$. *There exist matrices* $\mathbf{A}$ *of size* $(|\Lambda| - L) \times |\Lambda|$ *with* $\mathbf{A} \neq 0$ *such that there is no* $\mathbf{c} \in \mathbb{C}^L$ *such that the matrix*

$$\begin{bmatrix} \mathbf{G}(\mathbf{c})|_\Lambda \\ \mathbf{A} \end{bmatrix}$$

has full rank.

Theorem (Sufficient condition for identifiably)

Let $\mathbf{H} \in OPW(\Lambda)$ *with* $\Lambda \in \mathbb{Z}_L \times \mathbb{Z}_L$ *of size* $|\Lambda| = R > L$. *Assume that there exists a subset* $\widetilde{\Lambda} \subset \Lambda$ *of size* L *so that*

(i) $\tau_j(\Lambda) = \tau_j(\widetilde{\Lambda})$ *whenever* $\tau_j(\Lambda) \neq 0$.

(ii) $\operatorname{ind}(\tau') \neq \operatorname{ind}(\tau(\Lambda))$ *for every* L*-tubel* $\tau' \preceq \tau(\widetilde{\Lambda})$ *of size* L *different from* $\tau(\Lambda)$.

Given any full spark matrix $\mathbf{A}$ *of size* $(R-L) \times R$, *then there exists an identifier* $\mathbf{c} \in \mathbb{C}^L$ *so that the* $R \times R$ *matrix*

$$\begin{bmatrix} \mathbf{G}(\mathbf{c})|_\Lambda \\ \mathbf{A} \end{bmatrix}$$

is invertible. Moreover, the set of all such identifiers $\mathbf{c}$ *constitute a dense open subset of* $\mathbb{C}^L$.

Signal Transmission over Unidentified Channels

Motivation

- ▷ Two step procedure for data transmission over frequency-selective channels
  1. Estimate (identify) the channel $\mathbf{H} \in \mathscr{H} \subset \mathscr{L}(\mathbb{C}^L)$.
  2. Transmit data $\mathbf{x}$ from a certain data set $\mathscr{X} \subset \mathbb{C}^L$. Data recovery at the receiver using estimated channel.
- ▷ In time-varying channels, this procedure has to repeated regularly, to update channel state information.
- ▷ In rapidly changing channels, two step procedure becomes more and more inefficient.

Transmission through unidentified channel

- ▷ Combine channel identification and signal recovery.
- ▷ Transmission scheme $\mathbf{y} = \mathbf{H}(\mathbf{x} + \mathbf{c})$ with
  - $\mathbf{H} \in \mathscr{H} \subset \mathscr{L}(\mathbb{C}^L)$ is a unknown channel from a known subset $\mathscr{H}$.
  - $\mathbf{x} \in \mathscr{X}$ is the data signal from a certain data set $\mathscr{X} \subset \mathbb{C}^L$.
  - $\mathbf{c} \in \mathbb{C}^L$ is a pilot signal (designed based on the knowledge of $\mathscr{H}$ and $\mathscr{X}$).

Problem

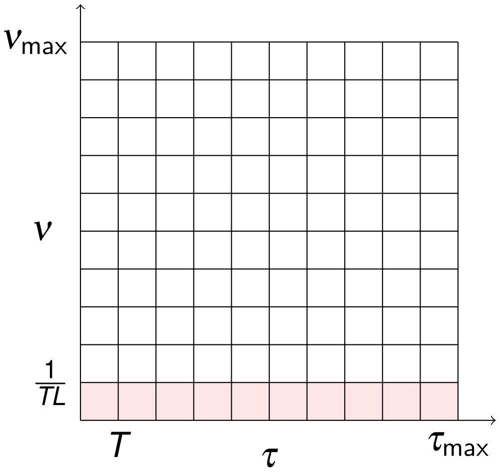
Find (necessary and/or sufficient) conditions on $\mathscr{H}$ and $\mathscr{X}$ such that there exists a pilot $\mathbf{c} \in \mathbb{C}^L$ such that every $\mathbf{x} \in \mathscr{X}$ can uniquely be recovered form $\mathbf{y} = \mathbf{H}(\mathbf{x} + \mathbf{c})$ for any unknown channel $\mathbf{H} \in \mathscr{H}$.

Relation to Blind Deconvolution

- $\mathscr{H} = OPW(\Lambda)$ with $\Lambda\{(0,0),(1,0),\dots,(L,0)\} \subset \mathbb{Z}_L \times \mathbb{Z}_L$

$$\mathbf{y} = \mathbf{Hc} = \sum_{k=0}^{L-1} \eta(k,0)\,\mathbf{T}^k\,\mathbf{x}$$

- Recover $\mathbf{x} \in \mathscr{X} \in \mathbb{C}^L$ and $\boldsymbol{\eta}$ from $\mathbf{y}$ (without knowing η).
- $\mathbf{x}$ and $\boldsymbol{\eta}$ assumed to be sparse.



Conditions for Data Recovery and/or Channel Identification

Natural Conditions

(I) Identifiability of $\mathscr{H}$: The map $\mathbf{H} \mapsto \mathbf{Hc}$ is injective on $\mathscr{H}$.

(R) Recovery condition for known channel: Every $\mathbf{H} \in \mathscr{H}$ is injective on $\mathscr{X}$.

Subsets of $\mathbb{C}^L$

$\mathscr{H}\mathbf{c} = \{\mathbf{Hc} \,:\, \mathbf{H} \in \mathscr{H}\}$: All possible output vectors for the pilot **c**.

$\mathscr{H}\mathscr{X} = \{\mathbf{Hx} \,:\, \mathbf{H} \in \mathscr{H}, \mathbf{x} \in \mathscr{X}\}$: Possible output of arbitrary data vector in $\mathscr{X}$.

Further Conditions

(i) $\operatorname{span}\{\mathscr{H}\mathbf{c}\} \cap \operatorname{span}\{\mathscr{H}\mathscr{X}\} = \{0\}$: Isolate **Hc** and **Hx** from the channel output $\mathbf{y} = \mathbf{H}(\mathbf{c}+\mathbf{x})$.

(ii) $\mathbf{H}(\mathscr{X}+\mathbf{c}) \cap \mathbf{H}'(\mathscr{X}+\mathbf{c})$ for every $\mathbf{H} \neq \mathbf{H}'$ in $\mathscr{H}$: Identify **H** from $\mathbf{y} = \mathbf{H}(\mathbf{c}+\mathbf{x})$ with unknown $\mathbf{x} \in \mathscr{X}$.

(iii) $\mathbf{H}(\mathbf{x}+\mathbf{c}) = \mathbf{H}'(\mathbf{x}'+\mathbf{c})$ implies $\mathbf{x} = \mathbf{x}'$: Guarantees exact recovery of $\mathbf{x} \in \mathscr{X}$ but not identification **H**

$$(i) \overset{\text{(I)}}{\Longrightarrow} (ii) \overset{\text{(R)}}{\Longrightarrow} (iii)$$

Degrees of Freedom

- $\mathscr{H} = OPW(\Lambda) \subset \mathscr{L}(\mathbb{C}^L)$ is a linear subspace of dimension $|\Lambda|$.
- Assume $\mathscr{X} \subset \mathbb{C}^L$ is a linear subspace of dimension K.
- Counting degrees of freedom, we must have

$$|\Lambda| + K \leq L \qquad \text{and} \qquad |\Lambda| \leq L\,. \tag{1}$$

  as a necessary condition for exact recovery of $\mathbf{H} \in \mathscr{H}$ and $\mathbf{x} \in \mathscr{X}$.
- Without identifying **H**, we may get $K > L - |\Lambda| \Longrightarrow$ how?
- When we get equality in (1)?

Example: 1-Dimensional Signal Space

- For a given (and known) $\Lambda \subset \mathbb{Z}_L \times \mathbb{Z}_L$, we consider the operator Paley–Wiener space

$$\mathscr{H} = OPW(\Lambda) = \Big\{ \mathbf{H} = \sum_{k,\ell \in \Lambda} \eta(k,\ell) \mathbf{M}^{\ell} \mathbf{T}^{k} \ :\ \eta(k,\ell) \in \mathbb{C} \Big\}. \subset \mathscr{L}(\mathbb{C}^L)$$

- Assume a 1–dimensional signal space $\mathscr{X} = \operatorname{span}\{\mathbf{v}\} \subset \mathbb{C}^L$ for some $\mathbf{v} \in \mathbb{C}^L$.
- For $x = u\mathbf{v} \in \mathscr{X}$, with $u \in \mathbb{C}$, the received signal is

$$\mathbf{y} = \mathbf{H}(\mathbf{x} + \mathbf{c}) = u\mathbf{H}(\mathbf{v}) + \mathbf{H}(\mathbf{c}) = \mathbf{G}(\mathbf{c})|_{\Lambda} \boldsymbol{\eta} + u\mathbf{G}(\mathbf{v})|_{\Lambda} \boldsymbol{\eta}$$

- Separation of **Hx** and **Hc** from **y** $\Longrightarrow \operatorname{span}\{\mathscr{H}\mathbf{c}\} \cap \operatorname{span}\{\mathscr{H}\mathbf{v}\} = \{0\} \Longrightarrow \operatorname{rang}[\mathbf{G}(\mathbf{c})|_{\Lambda}] \perp \operatorname{rang}[\mathbf{G}(\mathbf{v})|_{\Lambda}]$.

For $\operatorname{rang}[\mathbf{G}(\mathbf{v})|_{\Lambda}] = \operatorname{span}\{\mathbf{a}\}$ is a 1-dimensional subspace, we have the following result.

Theorem

Let $\Lambda \subset \mathbb{Z}_L \times \mathbb{Z}_L$ with $1 \leq |\Lambda| \leq L - 1$ and let $\mathbf{a} \in \mathbb{C}^L \backslash \{0\}$ be arbitrary.
There exists a vector $\mathbf{c} \in \mathbb{C}^L$ such that the $L \times |\Lambda| + 1$ matrix $[\mathbf{G}(\mathbf{c})|_{\Lambda}, \mathbf{a}]$ has full rank.

General Construction

- The set of all time-frequency shifts $\{\mathbf{M}^{\ell}\mathbf{T}^{k}\}_{\ell,k=0}^{L-1}$ can be separated into $L+1$ commutative subgroups

$$\mathscr{G}_s = \left\{\mathbf{M}^{2rk}\mathbf{T}^{k} : k = 0,1,\dots,L-1\right\}, \qquad s = 0,1,\dots L-1$$
$$\mathscr{G}_L = \left\{\mathbf{M}^{k} : k = 0,1,\dots,L-1\right\}$$

- Each commutative subgroup $\mathscr{G}_s$ posses a set of common eigenvectors (i.e. the *chirp sequences*)

$$\mathbf{e}_s(\ell) : \ell = 0,1,\dots,L-1$$

  which forms on orthogonal basis for $\mathbb{C}^L$.

Theorem

Let $\Lambda \subset \mathbb{Z}_L \times \mathbb{Z}_L$ such that there exists an $s \in \{0,1,\dots,L\}$ so that $\Lambda \subset \mathscr{G}_s$.
There exists a subspace $\mathscr{X} = \mathrm{span}\{\mathbf{e}_s(1),\dots,\mathbf{e}_s(K)\} \subset \mathbb{C}^L$ of dimension $K = L - |\Lambda|$ and a $\mathbf{c} \in \mathbb{C}^L$ such that

$$\mathrm{span}\{\mathscr{H}\mathbf{c}\} \cap \mathrm{span}\{\mathscr{H}\mathscr{X}\} = \{0\}$$

and such that Conditions (I) and (R) are satisfied.

Maximum Size of Data Subspace

- Given a subspace $\mathscr{X}$ of dimension K, it is desirable that the dimension of $\mathrm{span}\{\mathscr{H}\mathscr{X}\}$ is again K.
- For for a one-dimensional $\mathscr{X} = \mathrm{span}\{\mathbf{v}\}$, we have $\dim(\mathrm{span}\{\mathscr{H}\mathscr{X}\}) = \mathrm{rank}(\mathbf{G}(\mathbf{v})|_\Lambda)$.

Question: Given a support set $\Lambda \subset \mathbb{Z}_L \times \mathbb{Z}_L$ of $\mathscr{H} = OPW(\Lambda)$ with $|\Lambda| \leq L$.
What is the minimum rank of $\mathbf{G}(\mathbf{v})|_\Lambda$ for $\mathbf{v}$ varying in $\mathbb{C}^L$?

Theorem

Let $L \geq 3$ be an odd integer and $\Lambda \subset \mathbb{Z}_L \times \mathbb{Z}_L$ with $|\Lambda| \leq L$. Then

$$\min_{\mathbf{v} \in \mathbb{C}^L \setminus \{0\}} \mathrm{rank}(\mathbf{G}(\mathbf{v})|_\Lambda) \leq N(\Lambda)$$

with

$$N(\Lambda) = 1 + \min_{s \in \{0,1,\dots,L\}} \min\left\{ |I| \; : \; I \subset \mathbb{Z}_L \times \mathbb{Z}_L \text{ with } \Lambda \subset I + \mathscr{G}_s \right\},$$

and where $I + \mathscr{G}_s = \{x + y \; : \; x \in I, \; y \in \mathscr{G}_s\}$.

Example with Unknown Channel Support

Channel Model

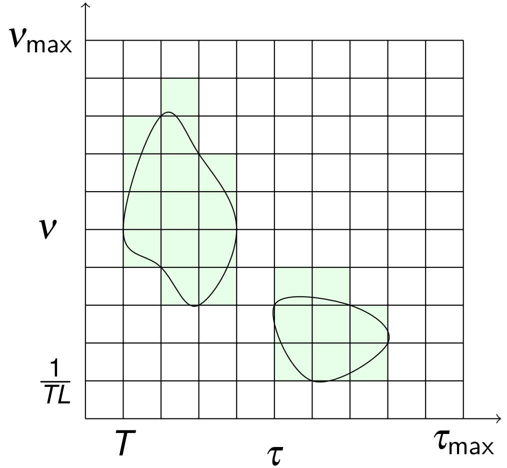
- Time continuous Time-Varying-Linear SISO Channel $\mathrm{H} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$:

$$g(t) = (\mathrm{H}f)(t) = \iint_{\mathbb{R}\times\mathbb{R}} \eta_{\mathrm{H}}(\tau,\nu) \cdot f(t-\tau)\,\mathrm{e}^{\mathrm{i}2\pi\nu t}\,\mathrm{d}\nu\,\mathrm{d}\tau = \iint_{\mathbb{R}\times\mathbb{R}} \eta_{\mathrm{H}}(\tau,\nu)\,(\mathrm{M}_\nu \mathrm{T}_\tau f)(t)\,\mathrm{d}\nu\,\mathrm{d}\tau$$

- Time discrete Time-Varying-Linear SISO Channel $\mathrm{H} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$:

$$g(t) = (\mathrm{H}f)(t) = \sum_{k=0}^{K-1}\sum_{\ell=0}^{M-1} \eta_{\mathrm{H}}(k,\ell) \cdot f(t-k\Delta\tau)\,\mathrm{e}^{\mathrm{i}2\pi\ell\Delta\nu t} = \sum_{k=0}^{K-1}\sum_{\ell=0}^{M-1} \eta_{\mathrm{H}}(k,\ell) \cdot f(t-kT)\,\mathrm{e}^{\mathrm{i}\frac{2\pi}{TL}\ell t}$$

- Rectification of the channel support region:
  with $\Delta\tau = T$ and $\Delta\nu = \frac{1}{TL}$ for some prime $L \geq 5$.



Transmitted Signal

- **Transmitted signal:** Delta train followed by a guard interval

$$f(t) = \sum_{m=0}^{2L-1} x_m \delta(t - mT) \qquad \text{with} \qquad x_m = \begin{cases} \text{data symbol} & : \ 0 \leq m \leq L-1 \\ 0 & : \ L \leq m \leq 2L-1 \end{cases}$$

- **Received signal:**
  1. Sampling at rate $1/T$: $g_n = (\mathrm{H}f)(nT), \ n = 0, 1, \dots, 2L-1.$
  2. Add two consecutive blocks of length L: $y_n = g_n + g_{n+L}, \ n = 0, 1, \dots, L-1.$
- **Write in vector form:**

$$\mathbf{y} = \mathbf{H}\mathbf{x} = \sum_{\ell=0}^{L-1} \sum_{k=0}^{L-1} \eta(k,l) \mathbf{M}^{\ell} \mathbf{T}^{k} \mathbf{x} \tag{2}$$

with $\mathbf{x} = (x_0, \dots, x_{L-1})^{\mathrm{T}}$ and $\mathbf{y} = (x_0, \dots, x_{L-1})^{\mathrm{T}}$ and with $\mathbf{H} \in OPW(\Lambda)$.

Remark: Using a periodic weighted delta train

$$f(t) = \sum_{p \in \mathbb{Z}} \sum_{m=0}^{2L-1} x_m \delta(t - mT - pLT)$$

results in a similar expression as in (2) but requires periodic data.

Transmission Scheme

- Aim: We want to transmit a message $\boldsymbol{\gamma} = \{\gamma_1, \dots, \gamma_Q\} \subset \mathbb{C}$ of size $Q \leq L$ over the channel $\mathbf{H} \in OPW(\Lambda)$ and recover $\boldsymbol{\gamma}$ at the receiver without knowing $\mathbf{H}$ and without knowing the channel support Λ in advance.
- Data symbols: The data symbols a sum of a pilot and the actual message

$$\mathbf{x} = \mathbf{c} + \sum_{q=1}^{Q} \gamma_q \, \mathbf{e}_q$$

with the pilot signal $\mathbf{c}$ which is chosen to be an *Alltop sequence* and $\mathbf{e}_q$ are particular *chirp sequences*

$$\mathbf{c}(n) = \frac{1}{\sqrt{L}} \exp\left(\mathrm{i}\frac{2\pi}{L} n^3\right) \quad \text{and} \quad \mathbf{e}_{mL+r}(n) = \frac{1}{\sqrt{L}} \exp\left(\mathrm{i}\frac{2\pi}{L}\left[r + mn + rn^2\right]\right), \qquad n \in \mathbb{Z}_L$$

- Received signal:

$$\mathbf{y} = \mathbf{H}\mathbf{x} = \sum_{\ell=0}^{L-1}\sum_{k=0}^{L-1} \eta(k,l)\,\mathbf{M}^{\ell}\mathbf{T}^{k}\mathbf{x} = \mathbf{G}(\mathbf{x})\boldsymbol{\eta} \; = \; \mathbf{G}(\mathbf{c})\boldsymbol{\eta} + \mathbf{U}\mathbf{s} = \boldsymbol{\Phi}\begin{pmatrix}\boldsymbol{\eta}\\ \mathbf{s}\end{pmatrix} \tag{3}$$

with a $L \times 2L^2$ measurement matrix $\boldsymbol{\Phi} = [\mathbf{G}(\mathbf{c})\ ,\ \mathbf{U}]$ and a sparse vector $\mathbf{s} = f(\boldsymbol{\eta}, \boldsymbol{\gamma})$.
- Recovery of the message: Solve CS problem (3), then determine $\boldsymbol{\gamma} = g(\boldsymbol{\eta}, \mathbf{s})$.

Compressive Sampling Problem

▷ Compressive Sampling Problem of size $L \times 2L^2$

$$\mathbf{y} = \mathbf{\Phi} \begin{pmatrix} \boldsymbol{\eta} \\ \mathbf{s} \end{pmatrix} \qquad \text{with} \qquad \left|\operatorname{supp}(\boldsymbol{\eta}, \mathbf{s})^{\mathrm{T}}\right| \leq (1+Q)\,|\Lambda|$$

Lemma

The coherence of the measurement matrix $\mathbf{\Phi} = [\mathbf{G}(\mathbf{c})\ ,\ \mathbf{U}]$ *is upper bounded by* $\mu(\mathbf{\Phi}) \leq \frac{2}{\sqrt{L}}$.

Remark: Welsh bound is $\frac{1}{\sqrt{L+1}}$.

Theorem

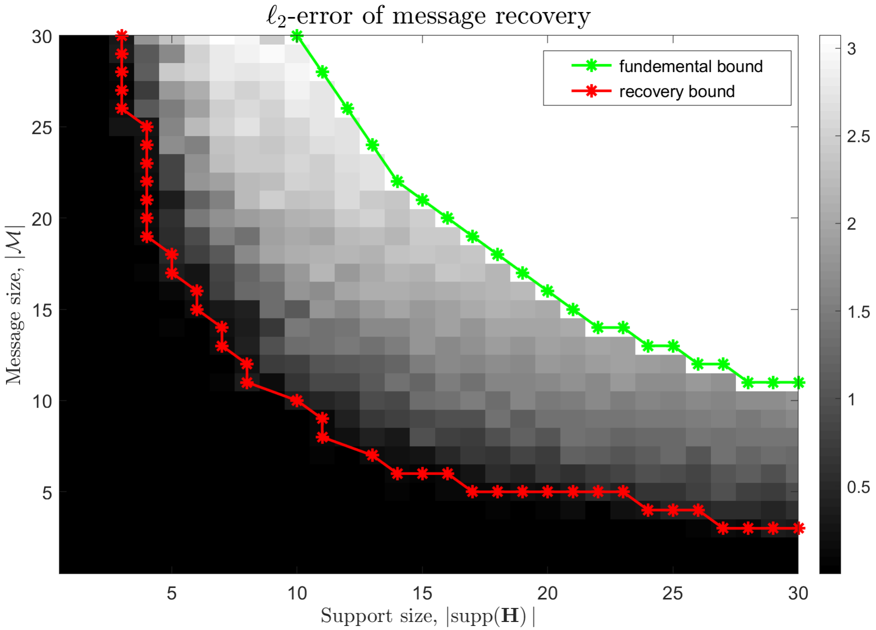
Let $\mathbf{H} \in OPW(\Lambda) \subset \mathscr{L}(\mathbb{C}^L)$ *be an unknown channel with unknown support set* Λ *where* $L \geq 5$ *is a prime number. For*

$$Q \leq \frac{\sqrt{L}}{4\,|\Lambda|} - 1$$

any message $\boldsymbol{\gamma} \in \mathbb{C}^Q$ *can be transmitted over* $\mathbf{H}$ *and recovered at the receiver.*

Numerical Experiment

- $L = 307$
- OMP
- average over 100 channels (random support, Gaussian coefficients)
- red line: 1% rel. error
- simulation much better than bound



Summary

Current and Future Work

- ▷ Identification of SISO and MIMO TVL Channels:
  - Identification of stochastic channels and stochastic sequences
  - Conditions on support of the covariance of the spreading function η.
  - Linear side constraints in terms of covariance.
- ▷ Transmission over Unidentified Channels:
  - Stochastic encoding, RIP
  - Scheme which maximum transmission rate
  - Continuous time setting

Related Publications

- D. G. Lee, G. E. Pfander, V. Pohl, W. Zhou "Identification of Channels with Single and Multiple Inputs and Outputs under Linear Constraints," *Linear Algebra Appl.*, vol. 581 (Nov. 2019), 435 – 470.
- D. G. Lee, G. E. Pfander, V. Pohl "Signal transmission through an unidentified channel," *13th Intern. Conf. on Sampling Theory and Applications (SampTA)*, Bordeaux, France, July 2019.
- A. Kaplan, D. G. Lee, V. Pohl "Message transmission through underspread time-varying linear channels," 45th Intern. Conf. on Acoustics, Speech, and Signal Processing (ICASSP), Barcelona, Spain, May 2020
- A. Kaplan, V. Pohl, D. G. Lee "The Statistical Restricted Isometry Property for Gabor Systems", *IEEE Statistical Signal Processing Workshop (SPP)*, Freiburg, Germany, June 2018, 45 – 49.
- D. G. Lee, G. E. Pfander, V. Pohl, W. Zhou "Identification of multiple-input multiple-output channels under linear side constraints‘," *43rd Intern. Conf. on Acoustics, Speech, and Signal Processing (ICASSP)*, Calgary, Canada, April 2018, 3889 – 3893.