

# MULTI-LEVEL COMPRESSED SENSING PETROV-GALERKIN DISCRETIZATION OF HIGH-DIMENSIONAL PARAMETRIC PDES

JEAN-LUC BOUCHOT, HOLGER RAUHUT, AND CHRISTOPH SCHWAB

ABSTRACT. We analyze a novel multi-level version of a recently introduced compressed sensing (CS) Petrov-Galerkin (PG) method from [H. Rauhut and Ch. Schwab: Compressive sensing Petrov-Galerkin approximation of high-dimensional parametric operator equations, *Math. Comp.* **304**(2017) 661–700] for the solution of many-parametric partial differential equations. We propose to use multi-level PG discretizations, based on a hierarchy of nested finite dimensional subspaces, and to reconstruct parametric solutions at each level from level-dependent random samples of the high-dimensional parameter space via CS methods such as weighted  $\ell_1$ -minimization. For affine parametric, linear operator equations, we prove that our approach allows to approximate the parametric solution with (almost) optimal convergence order as specified by certain summability properties of the coefficient sequence in a general polynomial chaos expansion of the parametric solution and by the convergence order of the PG discretization in the physical variables. The computations of the parameter samples of the PDE solution is “embarrassingly parallel”, as in Monte-Carlo Methods. Contrary to other recent approaches, and as already noted in [A. Doostan and H. Owhadi: A non-adapted sparse approximation of PDEs with stochastic inputs. *JCP* **230**(2011) 3015-3034] the optimality of the computed approximations does not require a-priori assumptions on ordering and structure of the index sets of the largest gpc coefficients (such as the “downward closed” property). We prove that under certain assumptions work versus accuracy of the new algorithms is asymptotically equal to that of one PG solve for the corresponding nominal problem on the finest discretization level up to a constant.

## 1. INTRODUCTION

Motivated in particular by uncertainty quantification, the numerical solution of parametric operator equations has gained significant attention in recent years. In many cases, the underlying parameter space is high dimensional or even infinite dimensional so that standard approximation methods are subject to the curse of dimensionality, see e.g. [19, 18]. Monte Carlo (*MC*) sampling, however, may be used in the context that the parametric model arises from a stochastic model and leads to a mean-square rate of  $m^{-1/2}$  in terms of the number  $m$  of sample evaluations, with constants that are independent of the parameter dimension. The (dimension-independent) rate  $1/2$  is not improvable in MC methods, in general, and the challenge consists in developing methods that achieve a faster convergence rate and at the same time alleviate or even overcome the curse of dimensionality.

A number of computational approaches have emerged in recent years towards this end. Among these are adaptive stochastic Galerkin methods, as developed in [28, 27, 36], reduced basis approaches (see, eg., [6, 14]), adaptive Smolyak discretizations [51, 52], adaptive interpolation methods [16] as well as sampling methods [54]. Adaptive Galerkin methods [28, 27, 36] are intrusive in the sense that they cannot simply reuse a solver developed for the corresponding problem with fixed parameter. In contrast, the other above mentioned methods and algorithms are non-intrusive, but they rely on successive numerical solutions of the operator equations for various parameter instances that are chosen based on suitable precomputations. In contrast, (multilevel) Monte-Carlo (*MLMC*) [46], or Quasi-Monte Carlo approaches (*QMC*) [25] compute expectations or statistical moments of the (random parametric) solution via solutions for parameter instances chosen at random or “quasi-random”, which allows to compute the “parameter snapshot” solutions in parallel.

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In this article, we build on a compressed sensing approach for numerically computing parametric solutions developed and analyzed in [48, 9] (see also [26, 47] for earlier work, and [17] for recent developments) and combine it with ideas originating from MLMC methods, see e.g. [35, 4]. For Petrov-Galerkin (PG) discretizations on a finite hierarchy of nested subspaces, ordered with respect to discretization levels, the presently proposed method “samples”, in a judicious fashion, the parameter space and computes the corresponding PG approximations for random choices of the parameter vector. As in MLMC-PG approaches, the number of such snapshot evaluations decreases with increasing discretization level (corresponding to increasing refinement of the discretization). In contrast to (ML)MC sampling, we employ a CS technique based on weighted  $\ell_1$ -minimization [49, 1] or iterative and greedy approaches (see for instance [42, 7, 33]) in order to reconstruct the coefficients of a generalized polynomial chaos expansion of the difference of the parametric solution at two subsequent discretization levels. Finally, these differences are summed together to obtain a PG approximation of the parametric solution at the finest level. One contribution of this paper is to show that the generalized polynomial chaos (GPC) expansion of the differences of PG approximations of the parametric solution is approximately sparse by estimating the weighted  $\ell_p$ -norm for  $0 < p < 1$  of the sequence of Chebyshev coefficients by a term that depends in a controlled way on the discretization level. This fact makes the presently developed, multi-level version of the compressive sensing approach feasible. We provide dimension-independent convergence rates which exceed  $1/2$  under certain sparsity assumptions on the parametric solution family of the operator equation and estimate the computational complexity for achieving such rates. Similar to MLMC methods, the workload for approximating the parametric solution is asymptotically the same as the one for computing one snapshot solution at the finest level up to a constant that depends only on smoothness parameters and  $p \in (0, 1)$ . However, in contrast to multilevel Monte Carlo, the convergence rates afforded by our scheme are practically independent of the dimension and only limited by the solutions’ sparsity; in particular, they may significantly exceed  $\mathcal{O}(m^{-1/2})$ .

In mathematical terms, we consider linear, parametric operator equations of the generic form

$$A(\mathbf{y})u(\mathbf{y}) = f. \quad (1)$$

Here the parameter vector  $\mathbf{y} \in U$  lies in a high-dimensional space  $U$  making it challenging to computationally approximate the solution map  $\mathbf{y} \mapsto u(\mathbf{y})$ , due to the mentioned *curse of dimensionality*, a notion going back to R.E. Bellman [5], see [19, 18] for its relevance in the present context. Assuming that the parameter vector  $\mathbf{y} = (y_j)_{j=1}^d$  takes values in finite intervals, we can consider, without loss of generality,  $U = [-1, 1]^d$ , where the parameter set dimension  $d$  may be finite or infinite.

In our setting, the parametric family of operators  $A(\mathbf{y}) : \mathcal{X} \rightarrow \mathcal{Y}'$  maps from a reflexive Banach space  $\mathcal{X}$  to the topological dual of, potentially, another reflexive Banach space  $\mathcal{Y}$ . A canonical example is the affine-parametric diffusion equation considered in [20, 21] and in the single-level version of the present work [48, 9]. For a bounded Lipschitz domain  $D \subset \mathbb{R}^n$  (one should think of  $n = 1, 2, 3$ ) and a parametric diffusion coefficient  $a(\cdot, \mathbf{y}) \in L^\infty(D)$  that depends affinely on a parameter vector  $\mathbf{y}$ , i.e.,

$$a(x, \mathbf{y}) = \bar{a}(x) + \sum_{j \geq 1} y_j \psi_j(x), \quad x \in D, \quad (2)$$

we consider the model parametric, second order divergence form elliptic Dirichlet problem

$$A(\mathbf{y})u := -\nabla \cdot (a(\cdot, \mathbf{y})\nabla u) = f \quad \text{in } D, \quad u|_{\partial D} = 0. \quad (3)$$

The weak formulation of (3) in the Sobolev space  $\mathcal{X} = \mathcal{Y} := H_0^1(D)$  reads: Given  $f \in \mathcal{Y}'$ , for every  $\mathbf{y} \in U := [-1, 1]^{\mathbb{N}}$  find  $u(\mathbf{y}) \in \mathcal{X}$  such that

$$\int_D a(x, \mathbf{y})\nabla u(x) \cdot \nabla v(x) dx = \int_D f(x)v(x) dx, \quad \text{for all } v \in \mathcal{Y}. \quad (4)$$

Eq. (3) is a particular example of an *affine-parametric operator equation* of the form

$$A(\mathbf{y}) := A_0 + \sum_{j \geq 1} y_j A_j, \quad \mathbf{y} = (y_j)_{j \geq 1} \in U := [-1; 1]^{\mathbb{N}}, \quad (5)$$

with  $A_j := -\nabla \cdot (\psi_j \nabla)$ ,  $A_0 = -\nabla \cdot (\bar{a} \nabla)$ . In (5), the operator  $A_0 \in \mathcal{L}(\mathcal{X}, \mathcal{Y}')$  is traditionally referred to as *nominal operator* or *mean field* while the operators  $A_j \in \mathcal{L}(\mathcal{X}, \mathcal{Y}')$ , for  $j \geq 1$ , are referred to as *fluctuations*. For the parametric problem to be well-posed uniformly with respect to the parameter  $\mathbf{y} \in U$ ,

we assume that  $\sum_{j \geq 1} \|A_j\|_{\mathcal{L}(X, Y')} < \infty$  in what follows. Further assumptions required for the convergence and applicability of our approach will be specified ahead. Parametric expansions such as (5) can be obtained e.g. by a Karhunen-Loève expansion of random input data for divergence-form partial differential equations, as explained in [53, 21].

In order to ensure well-posedness of the parametric diffusion problem (3) as in [21] we require the *uniform ellipticity assumption*: there exist constants  $0 < r \leq R < \infty$  such that

$$r \leq a(x, \mathbf{y}) \leq R, \quad \text{for almost all } x \in D, \text{ for all } \mathbf{y} \in U. \quad (6)$$

The Lax-Milgram Lemma ensures that for every  $\mathbf{y} \in U$ , the weak formulation (4) admits a unique solution  $u(\cdot, \mathbf{y}) \in \mathcal{X}$  which satisfies the uniform a priori estimate

$$\sup_{\mathbf{y} \in U} \|u(\mathbf{y})\|_{\mathcal{X}} \leq r^{-1} \|f\|_{Y'}.$$

Here and throughout the remainder, the term “uniform” refers to uniform with respect to the parameter sequence  $\mathbf{y} \in U$ .

For the sake of simplicity we detail here only the approximation of functionals  $\mathcal{G} \in \mathcal{X}'$  of solutions to the parametric operator equation (1), i.e., we are interested in the numerical approximation of

$$F(\mathbf{y}) := \mathcal{G}(u(\mathbf{y})), \quad \mathbf{y} \in U = [-1, 1]^d,$$

pointwise with respect to  $\mathbf{y}$ . We expect that our approach can be generalized to the recovery of the vector-valued solution map  $\mathbf{y} \mapsto u(\mathbf{y})$ , but we postpone this generalization to later contributions. We are aiming at numerical schemes that are:

- Reliable: the convergence and accuracy should be verified and customizable;
- Parallelizable: parallel sampling as in Monte-Carlo methods should be allowed, with a convergence rate in terms of the number of samples which (up to possibly logarithmic terms) equal the best possible rate ensured by the compressibility of  $F(\mathbf{y})$ , i.e., by weighted  $\ell_p$ -estimates of the Chebyshev coefficients of  $F$ ;
- Non-intrusive: the approximation should use existing numerical solvers of the problem with fixed parameters, without any re-implementation of PDE solvers.

It is important to notice the difference to usual MC methods where the results obtained from random sampling usually hold in expectation. In contrast, our approach provides approximations that hold pointwise with respect to  $\mathbf{y}$ . We estimate the coefficients of a tensorized Chebyshev expansion; whence only matrix-vector multiplications are required in order to compute the solution  $F(\mathbf{y}) = \mathcal{G}(u(\mathbf{y}))$  for any given parameter vector  $\mathbf{y} = (y_j)_{j=1}^d$  up to a prescribed accuracy. The computation scheme analyzed here differs from the single-level one introduced in [48] in the sense that computing the approximation is done in a more efficient and computationally tractable manner. To this end, an unknown function  $u(\mathbf{y})$  is approximated by a telescopic sequence of so-called “details” at successively finer spatial resolutions:  $u(\mathbf{y}) \approx \sum_{l=1}^L (u^l(\mathbf{y}) - u^{l-1}(\mathbf{y}))$  where  $u^l$  corresponds to a PG approximation on a discretization level  $l$ . This is analogous to MLMC methods, but is achieved here by *compressive sensing* of the parameters with a *level-dependent* number of parameter samples  $\mathbf{y}^{(i)}$  on each discretization level in the physical domain.

We outline key ideas of the compressive sensing approach. We assume at our disposal a countable orthonormal basis  $(\varphi_\nu)_{\nu \in \Lambda}$  of  $L^2(U, \eta)$  with  $\eta$  denoting a probability measure on the parameter set  $U$  to be specified, and denote by  $L^2(U, \eta; \mathcal{X})$  the Bochner space of strongly measurable maps from  $U$  to the (separable Hilbert) space  $\mathcal{X}$  containing solution instances, which are square integrable w.r.t.  $\eta$ . We represent any function  $u(\mathbf{y})$  with values in  $\mathcal{X}$  as  $u(\mathbf{y}) = \sum_{\nu \in \Lambda} \alpha_\nu \varphi_\nu(\mathbf{y})$ , where  $\alpha = (\alpha_\nu)_{\nu \in \Lambda}$  denotes the unique sequence of coefficients  $\alpha_\nu \in \mathcal{X}$ . Hence, in order to compute an approximation of the parametric solution for any  $\mathbf{y}$  it suffices to calculate an approximation of the coefficients  $\alpha_\nu$ . For a new input parameter  $\mathbf{y}$ , one evaluates the basis functions  $\varphi_\nu$  at  $\mathbf{y}$  and forms a linear combination to recover a direct estimation of the solution. Later on, we analyze the use of tensorized Chebyshev polynomials as orthonormal basis. The approximation is computed by evaluating the function at a few parameter points  $\mathbf{y}^{(i)}$ ,  $1 \leq i \leq m$ , and solving the linear system  $\mathbf{g} = \Phi \alpha$ , where  $\mathbf{g} = (g_i)_{i=1}^m = (u(\mathbf{y}^{(i)}))_{i=1}^m$  and where  $\Phi$  corresponds to the sensing matrix  $\Phi \in \mathbb{R}^{m \times N}$  with entries  $\Phi_{i,\nu} = \varphi_\nu(\mathbf{y}^{(i)})$ , where  $N$  corresponds to the number of basis functions taken for the approximation. However at this stage the coefficients  $\alpha_\nu$  and the components  $g_i$  are elements in  $\mathcal{X}$ , and therefore, we first

deal with the simpler case where a functional  $\mathcal{G}$  (also known as the Quantity of Interest ( $QoI$  for short) in the uncertainty quantification literature) is applied to the solution, resulting in

$$b_i = \mathcal{G}(g_i) = \mathcal{G}\left(u(\mathbf{y}^{(i)})\right) = \sum_{\nu \in \Lambda} z_\nu \varphi_\nu(\mathbf{y}^{(i)}), \quad i = 1, \dots, m, \quad z_\nu = \mathcal{G}(\alpha_\nu), \quad \nu \in \Lambda.$$

We are particularly interested in the situation that the number  $m$  of evaluations is smaller than the cardinality of  $\Lambda$ , so that the linear system  $b = \Phi z$  is underdetermined. Approximate sparsity of the coefficient sequence  $(\alpha_\nu)$ , and of  $(z_\nu)$ , allows to apply techniques from compressive sensing such as (weighted)  $\ell_1$ -minimization or iterative hard thresholding (pursuit) in order to recover  $z$  accurately. In fact, approximate sparsity follows from the fact that  $(\|\alpha_\nu\|_{\mathcal{X}})$  and  $(z_\nu)$  are contained in weighted  $\ell_p(\mathcal{F})$ -spaces, as shown in [3, 18, 20, 21, 48] and, for a related coefficient sequence, in this paper.

We expect that an approximation of the full solution  $u(\mathbf{y})$ ,  $\mathbf{y} \in U$ , taking values in the function space  $\mathcal{X}$ , can be computed by a variant of our compressive sensing scheme. One may use ideas from joint/block sparsity in order to recover the sequence  $(\alpha_\nu)$  with  $\alpha_\nu \in \mathcal{X}$  via mixed  $\ell_1/\ell_2$ -minimization, see e.g. [29, 30, 32] (at least in the case that  $\mathcal{X}$  and  $\mathcal{Y}$  are Hilbert spaces). However, we postpone a detailed analysis to a later contribution and restrict ourselves here to the simpler case of recovering the real-valued function  $\mathbf{y} \mapsto \mathcal{G}(u(\mathbf{y}))$ .

The multi-level approximation scheme uses discretization levels  $l = 1, \dots, L$ , where the meshwidth at discretization level  $l$  is  $2^{-l}h_0$ , so that the finest discretization is  $h_L = 2^{-L}h_0$ . With  $n$  being the dimension of the domain  $D$ , we assume that the number of degrees of freedom at level  $l$  scales like  $\mathcal{O}(2^{nl})$ , and we further assume *available linear complexity, multigrid solvers for the approximate solution of the discretized linear system of equations* (uniformly with respect to the parameter  $\mathbf{y}$ ) resulting in computational costs per PG solution  $u^l(\mathbf{y}^{(i)})$  that scales linearly in the number of degrees of freedom:  $\mathcal{O}(2^{nl})$ .

The presently proposed multi-level extension of the CS PG approach from [48] proceeds analogous to MLMC (see, e.g., [41] or [35] and the references therein): for parameter choices  $\{\mathbf{y}_l^{(i)}\}_{i=1, \dots, m_l}$  on the discretization level  $l$ , compute PG solutions  $u^l(\mathbf{y}_l^{(i)})$ ,  $u^{l-1}(\mathbf{y}_l^{(i)})$  at two consecutive discretization levels  $l$  and  $l-1$  (setting  $u^0 \equiv 0$ ). From the differences  $\Delta u^l(\mathbf{y}_l^{(i)}) = u^l(\mathbf{y}_l^{(i)}) - u^{l-1}(\mathbf{y}_l^{(i)})$ , we compute an approximation  $\widehat{\Delta u^l}(\mathbf{y})$  via the single level compressive sensing approach of [48] for each  $l = 1, \dots, L$ . Finally, we combine the approximations at all levels similarly as in MLMC methods, i.e.,  $u_{\text{MLCS}}^L(\mathbf{y}) = \sum_{l=1}^L \widehat{\Delta u^l}(\mathbf{y})$ , to obtain an approximation of the full parametric solution. The main result of this paper consists of an analysis of this method and provides, in its proof, a strategy on how to choose the number  $m_l$  of parameter points at each level  $l$ . Its precise statement, Theorem 9, is postponed to later in the exposition. To illustrate the type of results obtained here, we state now a version of Theorem 9 in the particular case of a linear, divergence form diffusion operator with affine dependence on the parameters (see Eqs. (2) and (3)). Ahead, we say that the weight sequence  $\mathbf{v}$  is constant, if it is of the form  $v_j = \beta$  for  $j = 1, \dots, d$  for some  $\beta > 1$  and  $v_j = \infty$  for  $j > d$ , which corresponds to the case that the expansion (2) is finite (with  $d$  terms). We say that  $\mathbf{v}$  has polynomial growth if  $v_j = cj^\alpha$ ,  $j \in \mathbb{N}$ , for some  $c > 1$ ,  $\alpha > 0$ . We refer to Section 5.3 for details on the weight sequences. Note that the order of the polynomial growth of the weight sequence does not affect the overall complexity (for a given target accuracy) of the method. It may however influence the multiplicative constants.

**Theorem 1.** *Let  $L \in \mathbb{N}$  and  $\gamma \in (0, 1)$ . Consider the diffusion equation (3) with affine parametric coefficient (2), forcing term  $f \in H^{-1+t}(D)$  and functional  $\mathcal{G} \in H^{-1+t'}(D)$ , with the respective smoothness parameters  $t, t' \geq 0$ . Assume that (2) holds with  $\bar{a} \in W^{t, \infty}(D)$  and that the fluctuations fulfill the weighted  $p$ -summability<sup>1</sup>*

$$\sum_{j \geq 1} \|\psi_j\|_{W^{t, \infty}(D)}^p v_j^{2-p} < \infty,$$

<sup>1</sup>To ease the presentation, here and throughout the paper, we have not highlighted the dependence of the summability parameter  $p$  on the regularity  $t$  of the right-hand-side  $f$ . It should be noted that the compressibility of the gpc expansion, the choice of the weight sequence, the number of samples per level all depend on the regularity of the data  $\bar{a}$ ,  $\psi_j$ ,  $D$  and  $f$ .

for a sequence  $\mathbf{v} = (v_j)_{j \geq 1}$  of weights as well as the following stronger, weighted version of the Uniform Ellipticity Assumption (6): there exists  $0 < r \leq R < \infty$  such that

$$\sum_{j \geq 1} v_j^{(2-p)/p} |\psi_j(x)| \leq \min\{\bar{a}(x) - r, R - \bar{a}(x)\}, \quad \text{for all } x \in D. \quad (7)$$

With probability at least  $1 - \gamma$ , the function  $F(\mathbf{y}) := \mathcal{G}(u(\mathbf{y}))$ ,  $\mathbf{y} \in U$ , can be approximated by  $L$  (weighted) sparse approximations (typically via weighted  $\ell^1$  minimization) based on a sequence of Galerkin projections into spaces of piecewise polynomials on regular, simplicial triangulations of meshwidth  $h_\ell = 2^{-\ell} h_0$  from

$$m_l \gtrsim \max\{s_l \log^3(s_l) \log(N_l), \log(L/\gamma)\}$$

solution evaluations at discretization level  $l$  for  $l = 1, \dots, L$ , where  $s_l \asymp 2^{(L-l)(t+t')p/(1-p)}$ , and  $N_l$  is the size of the (level-dependent) active set  $\Gamma_l$  of tensorized Chebyshev polynomials. The resulting approximation  $F^\#$  satisfies

$$\begin{aligned} \|F - F^\#\|_\infty &\leq C_p \|f\|_{H^{-1+t}(D)} \|\mathcal{G}\|_{H^{-1+t'}(D)} L 2^{-(t+t')L} h_0^{t+t'} \\ \|F - F^\#\|_2 &\leq C'_p \|f\|_{H^{-1+t}(D)} \|\mathcal{G}\|_{H^{-1+t'}(D)} 2^{-(t+t')L} h_0^{t+t'} \end{aligned}$$

Under the assumption that the computational cost of a single solve at level  $l$  scales linearly with respect to the number of degrees of freedom, i.e., is  $\mathcal{O}(2^{nl})$  (for an  $n$ -dimensional domain  $D$ ), this result is achieved with a total work for the computation of snapshot solutions that scales as  $\mathcal{O}\left(\max\left\{2^{nL}, L^\beta 2^{L(t+t')p/(1-p)}\right\}\right)$ , where  $\beta = 4$  for constant weights  $\mathbf{v}$  and  $\beta = 5$  for polynomially growing weights. The constant hidden in the  $\mathcal{O}$ -notation includes a factor of  $\log(d)$  in the case of constant weights.

We note in passing that in what follows, the estimates of the overall computational work do not account for the numerical solution of the (weighted) sparse approximation required for the compressed sensing estimation of the mapping  $F$ . We justify this convention by the observation that the computational cost of  $\ell_1$ -minimization is often of lower order compared to the total cost of evaluating the PDE samples. Section 6.2 validates empirically this claim.

Our theorem shows that in the case of sufficiently strong summability, i.e.,  $\frac{(t+t')p}{1-p} < n$ , at a total cost that scales as a constant times a single PDE solve at the finest discretization level  $L$ , the multilevel CSPG (MLCSPG) strategy can approximate a fixed function  $F$  for any parameter vector  $\mathbf{y} \in U$ . This is analogous to what is afforded by MLMC methods, but the present MLCSPG strategy allows to achieve any convergence rate afforded by the gpc summability, and allows to approximate the full parametric dependence, while MLMC only yields expectations (or moments). Moreover, in our case the computational work scales favorably with decreasing  $p$ , which corresponds to better sparse approximation rates implied by the weighted  $p$ -summability of (norms of) polynomial chaos coefficients of the parametric solution. To be more precise, in the case of higher smoothness  $t + t' > 0$ , we obtain an approximation error that scales with  $h_L^{t+t'}$ . With a small enough value of  $p$ , we may exploit smoothness in the physical domain (allowing  $t + t'$  such that  $\frac{(t+t')p}{1-p} < n$ ) and balance approximation error for the PDE solves. In contrast, the computational work required by MLMC to achieve an expected approximation error scaling as  $h_L^t$  grows proportionally to  $2^{2tL}$  when  $2t \geq n$  (where  $t$  corresponds to the smoothness of the solution in the physical domain), see [4, Theorem 5.7], and there is no parameter  $p$  in MLMC whose tuning allows to avoid such growth.

Nevertheless, we note that  $t$  and  $p$  may not be tuned independently: in many instances increased smoothness  $t$  leads to a larger value of the summability parameter  $p$ .

We emphasize that the tools and results developed here do not require a particular structure on the support set of the best approximation. It is often the case (see e.g. [16, 45]), that proofs and/or methods require the sets of active indices in  $N$ -term gpc approximations be *downward closed*, their approximation properties then being, in particular, independent of the polynomial system adopted for implementation. In contrast, the presently proposed, compressed sensing based approach can recover (with high probability) any support set of active multi-degrees of tensorized Chebyshev polynomial approximations (only assuming very rough knowledge of its location as provided by weighted  $\ell_p$ -estimates of polynomial chaos coefficients), yet still providing quasi-optimal rates of convergence. Moreover, apart from the  $\ell_1$ -minimization part of the algorithm, all function evaluations can be done in parallel. We would like to point out that while we do not

impose a particular structure on the coefficients, this structure is embedded in the choice of the coefficient sequence  $\omega$ . In particular, with the tensor product structure used in Eq. (52), one notices that the weights will favor indices  $\nu$  which have active components  $\nu_j > 0$  associated to smaller  $v_j$ . Setting smaller  $v_j$ 's will lead to a larger search space for the active coefficients  $\alpha_\nu$ . If the  $v_j$ 's scale inversely with the norm of the operators  $A_0^{-1}A_j$  and these operators are ordered in decreasing order of norm, then the choice (52) of  $\omega$  yields a downward close structure of the level-dependent sets  $\Gamma_l$  of active multi-indices. This downward closeness comes however as a consequence of the choice of weights, and not as a requirement for the method to work.

Theorem 1 is a particular case of our main Theorem 9 which we prove in Section 4 after recalling some basics about Petrov-Galerkin approximations in Section 2 combined with compressed sensing techniques in Section 3. Section 5 deals with practical aspects such as truncating the dimension of the parameter space. The paper is finally concluded by numerical experiments to illustrate the theory in Section 6.

## 2. PETROV-GALERKIN APPROXIMATIONS

We deal with the pointwise numerical approximations of the countably-parametric operator equation Eq. (1). Numerically accessing the parametric solution map  $U \ni \mathbf{y} \mapsto u(\mathbf{y})$  at a fixed parameter instance  $\mathbf{y} \in U$  requires discretization of Eq. (1) also in ‘‘physical space’’. To this end, we introduce two dense, one-parameter families of discretization spaces  $\{\mathcal{X}^h\}_{h>0} \subset \mathcal{X}$  and  $\{\mathcal{Y}^h\}_{h>0} \subset \mathcal{Y}$  of equal finite dimensions  $N^h := \dim(\mathcal{X}^h) = \dim(\mathcal{Y}^h)$  and assume that the parametric operator  $A(\mathbf{y})$  fulfills the discrete and uniform inf – sup conditions: there exists a  $\mu > 0$  such that for any  $h > 0$  and  $\mathbf{y} \in U$

$$\begin{cases} \inf_{0 \neq v^h \in \mathcal{X}^h} \sup_{0 \neq w^h \in \mathcal{Y}^h} \frac{\langle A(\mathbf{y})v^h, w^h \rangle}{\|v^h\|_{\mathcal{X}} \|w^h\|_{\mathcal{Y}}} \geq \mu > 0 \\ \inf_{0 \neq w^h \in \mathcal{Y}^h} \sup_{0 \neq v^h \in \mathcal{X}^h} \frac{\langle A(\mathbf{y})v^h, w^h \rangle}{\|v^h\|_{\mathcal{X}} \|w^h\|_{\mathcal{Y}}} \geq \mu > 0. \end{cases} \quad (8)$$

The PG projections are defined as the solution to the following weak variational problems:

$$\text{Find } u^h(\mathbf{y}) := G^h(\mathbf{y})(u(\mathbf{y})), \text{ such that } \langle A(\mathbf{y})u^h(\mathbf{y}), v^h \rangle = \langle f, v^h \rangle \quad \text{for all } v^h \in \mathcal{Y}^h. \quad (9)$$

We recall the following classical result (see for example [8, Chapter 6]).

**Proposition 1.** *Let  $\mathcal{X}^h$  and  $\mathcal{Y}^h$  be discretization spaces for the PG method, such that the uniform discrete inf – sup conditions (8) are fulfilled and assume that the bilinear operator  $\mathcal{X} \times \mathcal{Y} \ni (u, w) \mapsto \langle A(\mathbf{y})u, w \rangle$  is continuous, uniformly with respect to  $\mathbf{y} \in U$ .*

*Then the PG projections  $G^h(\mathbf{y}) : \mathcal{X} \rightarrow \mathcal{X}^h$  are well-defined linear operators, whose norms are uniformly bounded with respect to the parameters  $\mathbf{y}$  and  $h$ , i.e.,*

$$\sup_{\mathbf{y} \in U} \sup_{h>0} \|u^h(\mathbf{y})\|_{\mathcal{X}} \leq \frac{1}{\mu} \|f\|_{\mathcal{Y}}, \quad (10)$$

$$\sup_{\mathbf{y} \in U} \sup_{h>0} \|G^h(\mathbf{y})\|_{\mathcal{L}(\mathcal{X})} \leq \frac{C}{\mu} \quad (11)$$

*The Galerkin projections are uniformly quasi-optimal: for every  $\mathbf{y} \in U$  we have the a-priori error bound*

$$\|u(\mathbf{y}) - u^h(\mathbf{y})\|_{\mathcal{X}} \leq \left(1 + \frac{C}{\mu}\right) \inf_{v^h \in \mathcal{X}^h} \|u(\mathbf{y}) - v^h\|_{\mathcal{X}}. \quad (12)$$

As is classical in the theory of polynomial approximation (see, e.g. [22, 50]), we use a holomorphic extension of the parametric operator family  $A(\mathbf{y})$  to complex parameter sequences  $\mathbf{z} \in \mathcal{O} \supset U$ , where  $\mathcal{O}$  is some suitable subset of the complex plane. Here, when dealing with extensions of operators and solutions to parameters taking values in the complex domain, we identify the function spaces  $\mathcal{X}$  and  $\mathcal{Y}$  with their complexifications  $\mathcal{X} \otimes \{1, \mathbf{i}\}$  and  $\mathcal{Y} \otimes \{1, \mathbf{i}\}$  for the sake of simplicity. We require the parametric operator  $\mathcal{O} \ni \mathbf{z} \mapsto A(\mathbf{z})$  to be holomorphic with respect to any finite set of variables and to be boundedly invertible. Hereby, a Banach-space valued mapping  $z \mapsto R(z) \in E$  of a single complex variable is said to be holomorphic (in some open domain  $\mathcal{O}$ ) if

$$\lim_{h \rightarrow 0} \frac{R(z_0 + h) - R(z_0)}{h}$$

exists in  $E$  for any  $z_0 \in \mathcal{O}$ , with  $h \rightarrow 0$  understood in  $\mathbb{C}$ . Note that our assumption on  $A$  requires it to be holomorphic with respect to any component  $z_j$  of  $\mathbf{z}$  independently. Joint holomorphy with respect to an arbitrary, finite subset of variables  $\mathbf{z}' = (z_j)_{j \in \Lambda}$  with  $|\Lambda| < \infty$  of  $\mathbf{z}$  then follows from Hartogs' theorem. In the sequel, we will often assume that the open set  $\mathcal{O}$ , on which  $\mathbf{z} \mapsto A(\mathbf{z})$  is holomorphic, contains the product of Bernstein ellipses  $\mathcal{E}_\rho = \bigotimes_{j \geq 1} \mathcal{E}_{\rho_j}$  with  $\mathcal{E}_\sigma := \{(z + z^{-1})/2, z \in \mathbb{C} : |z| = \sigma\}$ . In the case of complex-parametric operators, the bounded invertibility of  $A(\mathbf{z})$  is equivalent to the *complex discrete inf – sup conditions*: there exists a constant  $\mu_C > 0$  such that for any  $h > 0$  and  $\mathbf{z} \in \mathcal{O}$

$$\begin{cases} \inf_{0 \neq v^h \in \mathcal{X}^h} \sup_{0 \neq w^h \in \mathcal{Y}^h} \operatorname{Re} \frac{\langle A(\mathbf{z})v^h, w^h \rangle_{\mathcal{Y}}}{\|v^h\|_{\mathcal{X}} \|w^h\|_{\mathcal{Y}}} \geq \mu_C > 0, \\ \inf_{0 \neq w^h \in \mathcal{Y}^h} \sup_{0 \neq v^h \in \mathcal{X}^h} \operatorname{Re} \frac{\langle A(\mathbf{z})v^h, w^h \rangle_{\mathcal{Y}}}{\|v^h\|_{\mathcal{X}} \|w^h\|_{\mathcal{Y}}} \geq \mu_C > 0. \end{cases} \quad (13)$$

Approximation results on discretization spaces are usually combined with prior knowledge of the regularity of the data. For this, we assume that there exists a  $0 < t \leq \bar{t}$  such that the parametric family  $A(\mathbf{y}) \in \mathcal{L}(\mathcal{X}, \mathcal{Y}')$  is regular in given smoothness scales  $\{\mathcal{X}_t\}_{t \geq 0}$ , resp.  $\{\mathcal{Y}_t\}_{t \geq 0}$ , satisfying:

$$\mathcal{X} = \mathcal{X}_0 \supset \mathcal{X}_1 \supset \dots \supset \mathcal{X}_t, \quad \mathcal{Y} = \mathcal{Y}_0 \supset \mathcal{Y}_1 \supset \dots \supset \mathcal{Y}_t. \quad (14)$$

Here, the smoothness index  $t \geq 0$  denotes, for example, a differentiation order in a scale of Sobolev or Besov spaces. These spaces are defined by interpolation for non integer indices. We shall also require a corresponding scale on the dual side, with  $\mathcal{X}'_t := (\mathcal{X}')_t$ , and  $\mathcal{Y}'_t := (\mathcal{Y}')_t$ :

$$\mathcal{X}' = \mathcal{X}'_0 \supset \mathcal{X}'_1 \supset \dots \supset \mathcal{X}'_t, \quad \mathcal{Y}' = \mathcal{Y}'_0 \supset \mathcal{Y}'_1 \supset \dots \supset \mathcal{Y}'_t. \quad (15)$$

Note carefully that with this notation,  $(\mathcal{X}_t)'$  **generally differs** from  $\mathcal{X}'_t$ . For example, in the case of the diffusion equation (3), one may choose  $\mathcal{X} = H_0^1(D)$  and  $\mathcal{X}_t = H_0^{1+t}(D)$ . In this case,  $\mathcal{X}' = H^{-1}(D) = (H_0^1(D))'$  and  $(\mathcal{X}_t)' = H^{-1-t}(D) \neq H^{-1+t}(D) = (\mathcal{X}'_t) =: \mathcal{X}'_t$ .

A first statement of solution regularity in the scales (14), (15) takes the form of uniform bounded invertibility of the family of parametric operators  $A(\mathbf{y})$ :

$$A(\mathbf{y}) \in \mathcal{L}(\mathcal{X}_t, \mathcal{Y}'_t), \quad \text{for all } \mathbf{y} \in U, \quad \text{and } \sup_{\mathbf{y} \in U} \|A(\mathbf{y})^{-1}\|_{\mathcal{L}(\mathcal{Y}'_t, \mathcal{X}_t)} < \infty. \quad (16)$$

For the PG discretization, we assume at hand two one-parameter families  $\{\mathcal{X}^h\}_{h>0}$  and  $\{\mathcal{Y}^h\}_{h>0}$  of  $\mathcal{X}$  and  $\mathcal{Y}$ , respectively, with finite, equal dimension:  $N^h = \dim(\mathcal{X}^h) = \dim(\mathcal{Y}^h) < \infty$ . We assume furthermore that  $\{\mathcal{X}^h\}_{h>0}$  and  $\{\mathcal{Y}^h\}_{h>0}$  are dense in  $\mathcal{X}$  and in  $\mathcal{Y}$ , respectively. Here the discretization parameter  $h > 0$  usually stands for the meshwidth in finite element discretizations of fixed polynomial degree, on a quasiuniform triangulation of the physical bounded, polyhedral domain  $D$ . We assume that these spaces admit *linear approximation properties* in the smoothness scales<sup>2</sup> (14), (15),

$$\begin{aligned} \inf_{w^h \in \mathcal{X}^h} \|w - w^h\|_{\mathcal{X}} &\leq C_t h^t \|w\|_{\mathcal{X}_t}, & \text{for all } w \in \mathcal{X}_t, \\ \inf_{v^h \in \mathcal{Y}^h} \|v - v^h\|_{\mathcal{Y}} &\leq C_{t'} h^{t'} \|v\|_{\mathcal{Y}_{t'}}, & \text{for all } v \in \mathcal{Y}_{t'}. \end{aligned} \quad (17)$$

Such approximation properties hold, for example, for the model Dirichlet problem (3) in polytopal domain  $D \subset \mathbb{R}^d$  and for Lagrangian Finite Elements of polynomial degree  $p \geq 1$  on quasiuniform, regular simplicial triangulations of  $D$  of meshwidth  $h$  with the choices  $\mathcal{X} = \mathcal{Y} = H_0^1(D)$  and  $\mathcal{X}_t = \mathcal{Y}_t = H^{1+t}(D) \cap H_0^1(D)$  for  $0 \leq t, t' < \min\{t^*, p\}$ . Here,  $t^* > 0$  denotes a limit on isotropic Sobolev regularity of the solution of (3) in  $D$  which is due to several factors: a) smoothness of  $\partial D$ , b) smoothness of  $f$  and c) smoothness of the parametric coefficient  $\mathbf{y} \mapsto a(\cdot, \mathbf{y})$ . Alternative choices (with possibly larger ranges of  $t^*$ ) are weighted (Kondrat'ev) spaces  $\mathcal{X}_t = \mathcal{Y}_t$  and Lagrangean Finite Elements of polynomial degree  $p \geq 1$  on locally refined regular simplicial triangulations of  $D$  of meshwidth  $h$ .

Together with the bounded invertibility of the family of operators  $A(\mathbf{y})$ , it holds:

$$\|u(\mathbf{y}) - u^h(\mathbf{y})\|_{\mathcal{X}} \stackrel{\text{Eq. (12)}}{\leq} C \inf_{v^h \in \mathcal{X}^h} \|u(\mathbf{y}) - v^h\|_{\mathcal{X}} \stackrel{\text{Eq. (17)}}{\leq} c_t h^t \|u(\mathbf{y})\|_{\mathcal{X}_t} \stackrel{\text{Eq. (16)}}{\leq} C_t h^t \|f\|_{\mathcal{Y}'_t}.$$

<sup>2</sup>Note that it would be possible to include the case of higher order FEM by propagating the order  $k$  of FEM in the remaining parts of the estimations. We have chosen not to derive these results here to ease the presentation.

Here, the constant  $C_t$  depends on a uniform bound on the inverse of the parametric operator in the appropriate smoothness space:  $\sup_{\mathbf{y} \in U} \|A(\mathbf{y})^{-1}\|_{\mathcal{L}(\mathcal{Y}'_t, \mathcal{X}_t)}$ , and on the smoothness parameter  $t$ , but not on the discretization parameter  $h$ .

Moreover, as we confine the exposition to functionals of solutions  $F(\mathbf{y}) = \mathcal{G}(u(\mathbf{y}))$  for some  $\mathcal{G}(\cdot) \in \mathcal{X}'$ , we assume *adjoint regularity*, i.e., there exists  $t' \geq 0$ , such that  $\mathcal{G} \in \mathcal{X}'_{t'}$ , and such that the parametric adjoint solution  $w_{\mathcal{G}}(\mathbf{y}) \in \mathcal{Y}'$  of the problem

$$A(\mathbf{y})^* w_{\mathcal{G}}(\mathbf{y}) = \mathcal{G} \quad (18)$$

satisfies  $w_{\mathcal{G}}(\mathbf{y}) \in \mathcal{Y}'_{t'}$  uniformly with respect to  $\mathbf{y}$ :

$$\sup_{\mathbf{y} \in U} \|w_{\mathcal{G}}(\mathbf{y})\|_{\mathcal{Y}'} \leq C \|\mathcal{G}\|_{\mathcal{X}'_{t'}}. \quad (19)$$

Under the adjoint regularity (19), the uniform parametric discrete inf-sup condition (8) and the approximation property (17), an Aubin-Nitsche duality argument as, e.g., in [43], implies superconvergence: for any  $\mathbf{y} \in U$ , with  $F^h$  the functional applied to the parametric PG solution  $u^h(\mathbf{y})$  defined in (9) on discretization spaces of parameter  $h$ ,

$$|F(\mathbf{y}) - F^h(\mathbf{y})| \leq C_{t+t'} h^{t+t'} \|f\|_{\mathcal{Y}'_t} \|\mathcal{G}\|_{\mathcal{X}'_{t'}}. \quad (20)$$

### 3. SINGLE-LEVEL COMPRESSED SENSING PETROV-GALERKIN APPROXIMATIONS

The multi-level compressed sensing PG (MLCSPG) discretization is a generalization of the single-level algorithms and results developed in [48]. Analogous to MLMC path simulations (see e.g. [35] and the references there) or MLMC Finite Element discretizations (see e.g. [4]) the MLCSPG method described here considers a sampling scheme from [48] with a number of sampling points depending on the discretization level.

Such compressed sensing reconstruction techniques have already shown promise in the context of numerical solutions of PDEs on high-dimensional parameter spaces: we refer, for example, to [57, 26, 47, 48, 9]. Note that these approaches differ from other compressed-sensing based approaches that are used for efficiently computing a single snapshot, see for instance [12, 13]. In their work, the authors do not use weighted versions of compressed sensing and only use compressibility in the spatial domain. In comparison, our work considers a compression in the parameter space, and a recovery using weighted compressed sensing.

The key idea in the works [57, 26, 47, 48, 9] is to decompose the solution of Eq. (1) via its (tensorized Chebyshev or Legendre) polynomial chaos expansion with respect to the parameter vector  $\mathbf{y}$ . A strongly measurable mapping  $u : U \rightarrow \mathcal{X} : \mathbf{y} \mapsto u(\mathbf{y})$  which is square (Bochner-) integrable with respect to the Chebyshev measure  $d\eta$  over  $U$  can be represented as a gpc expansion, i.e.,

$$u(\mathbf{y}) = \sum_{\nu \in \mathcal{F}} u_{\nu} T_{\nu}(\mathbf{y}), \quad (21)$$

where in this case the coefficients in this expansion are functions  $u_{\nu} \in \mathcal{X}$ . Here  $\mathcal{F} := \{\nu \in \mathbb{N}_0^{\mathbb{N}} : |\text{supp}(\nu)| < \infty\}$  is the set of multi-indices with finite support. The tensorized Chebyshev polynomials are defined as

$$T_{\nu}(\mathbf{y}) = \prod_{j=1}^{\infty} T_{\nu_j}(y_j) = \prod_{j \in \text{supp}(\nu)} T_{\nu_j}(y_j), \quad \mathbf{y} \in U, \quad \nu \in \mathcal{F}, \quad (22)$$

with the univariate Chebyshev polynomials defined by

$$T_j(t) = \sqrt{2} \cos(j \arccos(t)), \quad \text{and } T_0(t) \equiv 1. \quad (23)$$

Defining the probability measure  $\sigma$  on  $[-1; 1]$  as  $d\sigma(t) := \frac{dt}{\pi\sqrt{1-t^2}}$ , the univariate Chebyshev polynomials  $T_j$  defined in (23) form an orthonormal system in  $L^2([-1, 1]; \sigma)$  in the sense that

$$\int_{-1}^1 T_k(t) T_l(t) d\sigma(t) = \delta_{k,l}, \quad k, l \in \mathbb{N}_0.$$

Similarly, with the product measure

$$d\eta(\mathbf{y}) := \bigotimes_{j \geq 1} d\sigma(y_j) = \bigotimes_{j \geq 1} \frac{dy_j}{\pi\sqrt{1-y_j^2}},$$

the tensorized Chebyshev polynomials (22) are orthonormal with respect to  $\eta$  in the sense that

$$\int_{\mathbf{y} \in U} T_\mu(\mathbf{y}) T_\nu(\mathbf{y}) d\eta(\mathbf{y}) = \delta_{\mu,\nu}, \quad \mu, \nu \in \mathcal{F}.$$

A result proven in [39] ensures the  $\ell_p$  summability, for some  $0 < p \leq 1$ , of the polynomial chaos expansion (21) for the diffusion case, Eq. (3):

$$\left\| (\|u_\nu\|_{\mathcal{X}})_{\nu \in \mathcal{F}} \right\|_p^p = \sum_{\nu \in \mathcal{F}} \|u_\nu\|_{\mathcal{X}}^p < \infty$$

under the uniform ellipticity assumption (6) and the condition that the sequence of infinity norms of the  $\psi_j$  is itself  $\ell_p$  summable:

$$\left\| (\|\psi_j\|_\infty)_{j \geq 1} \right\|_p^p = \sum_{j \geq 1} \|\psi_j\|_\infty^p < \infty.$$

Recent results by [3] show that these conditions can be improved by considering pointwise convergence of the series  $\sum_{j \geq 1} |\psi_j|$  instead of infinity norms in the whole domain  $D$ . This takes advantage of the local structure of the basis elements  $\psi_j$ , e.g. when only few of them are overlapping, as is the case for wavelets. In particular,  $\ell_p$  summability of Legendre and Chebyshev coefficients can be obtained when  $(\|\psi_j\|_\infty)_j \in \ell_q$  for  $q := 2p/(2-p)$  provided that the interiors of the supports of the  $\psi_j$  do not overlap. The summability results from [39] concerning Chebyshev expansions were extended to weighted  $\ell_p$  estimates for the general parametric operator problem (1) with affine dependence as in (5), in [48] under slightly stronger assumptions. This result is particularly important for us as it ensures the recovery of the coefficients  $u_\nu$  (or any functional thereof) via CS methods.

The results on the approximation via an MLCSPG framework rely on the single-level results developed in [49, 48], where functions are approximated via a weighted-sparse expansion in an appropriate basis. We review here the main ideas. Given a (finite) orthonormal system  $(\phi_\nu)_{\nu \in \Lambda}$ , with  $|\Lambda| = N < \infty$  for  $L^2(U, \eta)$  where  $\eta$  is a probability measure, for any fixed function  $f : U \rightarrow \mathbb{R}$ , there exists a unique sequence of coefficients  $\mathbf{f} = (f_\nu)_{\nu \in \Lambda}$  such that

$$f(\mathbf{y}) = \sum_{\nu \in \Lambda} f_\nu \phi_\nu(\mathbf{y}), \quad \forall \mathbf{y} \in U. \quad (24)$$

We define an  $\ell_p$  norm associated with this expansion as  $\|f\|_p := \|\mathbf{f}\|_p$ .

In particular, a function  $f$  is said to be sparse (or compressible) if its sequence of coefficients in expansion (24) is sparse (or compressible) itself. Our goal is to recover this said sequence from seemingly few evaluations of the function  $f$  at certain (here random) sample points  $\mathbf{y}^{(i)}$ , for  $1 \leq i \leq m$ . This can be done by CS methods: after introducing the sensing matrix  $\Phi$  as  $\Phi_{i,j} := \phi_j(\mathbf{y}^{(i)})$  and letting  $\mathbf{g}_i := f(\mathbf{y}^{(i)})$ , it holds

$$\mathbf{g} = \Phi \mathbf{f}.$$

Hence, assuming that the expansion is sparse, and the number of samples rather small, we are dealing with the by-now classical problem of recovering a sparse vector from few linear measurements, by solving, for instance, the convex program

$$\min_{\mathbf{z} \in \mathbb{R}^N} \|\mathbf{z}\|_1, \quad \text{subject to } \Phi \mathbf{z} = \mathbf{g}. \quad (25)$$

In our context, it is beneficial to use a *weighted framework* which has been developed recently in [49]. Introducing a sequence of positive weights  $(\omega_\nu)_{\nu \in \Lambda}$  with  $|\omega_\nu| \geq 1$  for all  $\nu$ , a weighted  $\ell_p$  (quasi-)norm (henceforth indexed  $\ell_{\omega,p}$  when appropriate) can be defined as

$$\|\mathbf{f}\|_{\omega,p}^p := \sum_{\nu \in \Lambda} \omega_\nu^{2-p} |f_\nu|^p, \quad 0 < p \leq 2.$$

In particular, it holds  $\|\mathbf{f}\|_{\omega,2} = \|\mathbf{f}\|_2$  and  $\|\mathbf{f}\|_{\omega,1} = \|\mathbf{f} \odot \omega\|_1$ , where  $\odot$  defines the pointwise multiplication. Moreover, choosing the constant weight  $\omega_\nu = 1$  yields the original definitions of  $\ell_p$  norms. Formally letting  $p \downarrow 0$  motivates the introduction of the *weighted sparsity* measure

$$\|\mathbf{f}\|_{\omega,0} := \sum_{\nu \in \Lambda, f_\nu \neq 0} \omega_\nu^2.$$

A vector  $\mathbf{x}$  is therefore called *weighted  $s$ -sparse* (with respect to a weight sequence  $\omega$ ) if  $\|\mathbf{x}\|_{\omega,0} \leq s$ . We may therefore define the error of best weighted  $s$ -term approximation as

$$\sigma_{\omega,s}(f) = \sigma_{\omega,s}(\mathbf{f}) := \inf_{\mathbf{z}: \|\mathbf{z}\|_{\omega,0} \leq s} \|\mathbf{f} - \mathbf{z}\|_{\omega,p}.$$

With these weighted error measures at hand, the Basis Pursuit problem (25) can be generalized to include a-priori information encoded in the sequences  $\omega$  of weights, as

$$\min_{\mathbf{z} \in \mathbb{R}^N} \|\mathbf{z}\|_{\omega,1}, \quad \text{subject to } \Phi \mathbf{z} = \mathbf{g}. \quad (26)$$

More details on such weighted spaces and weighted sparse approximations can be found in [49] where the following fundamental result is also proved.

**Theorem 2.** *Suppose  $(\phi_\nu)_{\nu \in \Lambda}$  is a finite orthonormal system with  $|\Lambda| = N < \infty$  and that weights  $\omega_\nu \geq \|\phi_\nu\|_\infty$  are given. For a (weighted) sparsity  $s \geq 2\|\omega\|_\infty^2$ , draw*

$$m \geq Cs \max\{\log^3(s) \log(N), \log(1/\gamma)\} \quad (27)$$

sample points  $\mathbf{y}^{(i)}$  at random,  $1 \leq i \leq m$ , according to the orthonormalization measure  $\eta$ . The constant  $C > 0$  in (27) is universal, i.e., independent of all other quantities including  $s$ ,  $m$  and  $N$ .

Then, with probability at least  $1 - \gamma$ , any function  $f = \sum f_\nu \phi_\nu$  can be approximated by the function  $\hat{f} := \sum \hat{f}_\nu \phi_\nu$ , where  $\hat{\mathbf{f}}$  is the solution to the weighted basis pursuit problem (26). The approximation holds in the following sense:

$$\|f - \hat{f}\|_\infty \leq \left\| \|f - \hat{f}\|_{\omega,1} \right\| \leq c_1 \sigma_s(f)_{\omega,1}, \quad \text{and} \quad \|f - \hat{f}\|_2 \leq d_1 \sigma_s(f)_{\omega,1} / \sqrt{s}.$$

In particular, using the weighted Stechkin inequality from [49]

$$\sigma_s(\mathbf{f})_{\omega,q} \leq (s - \|\omega\|_\infty^2)^{1/q-1/p} \|\mathbf{f}\|_{\omega,p}, \quad p < q \leq 2, \quad \|\omega\|_\infty^2 < s, \quad (28)$$

we obtain that for given summability exponent  $0 < p < 1$ , there exists a constant  $C > 0$  independent of  $s$  such that

$$\|f - \hat{f}\|_\infty \leq Cs^{1-1/p} \|\mathbf{f}\|_{\omega,p}, \quad \|f - \hat{f}\|_2 \leq Cs^{1/2-1/p} \|\mathbf{f}\|_{\omega,p}. \quad (29)$$

Choosing  $m \asymp s \ln^3(s) \log(N)$  relates the reconstruction error and the number  $m$  of samples as

$$\|f - \hat{f}\|_\infty \leq c \left( \frac{\log^3(m) \log(N)}{m} \right)^{1/p-1} \|\mathbf{f}\|_{\omega,p}, \quad \|f - \hat{f}\|_2 \leq d \left( \frac{\log^3(m) \log(N)}{m} \right)^{1/p-1/2} \|\mathbf{f}\|_{\omega,p}.$$

**Remark 1.** *Recent works [11, 40] on restricted isometry constants for subsampled Fourier matrices suggest that the factor  $\log^3(s)$  in (27) can be reduced to  $\log^2(s)$ .*

#### 4. MULTI-LEVEL COMPRESSED SENSING PETROV-GALERKIN APPROXIMATIONS

**4.1. A multi-level framework.** We extend the foregoing CS methods to sweeping the parameter domain to multi-level (“ML” for short) discretizations of the parametric problems, in the spirit of the ML MC methods for numerical treatment of operator equations with random inputs as developed in [41, 35, 4]. There, the solution of the parametric operator equation (1) is approximated on a sequence of partitions of the physical domain  $D$  of widths  $\{h_l\}_{l=1}^L$  for a prescribed, maximal refinement level  $L \in \mathbb{N}$ . To simplify the exposition, we assume dyadic refinement, i.e.  $h_{l+1} = h_l/2 = 2^{-l}h_0$  for a given, small enough, initial resolution  $h_0 > 0$ .

For a given parameter sequence  $\mathbf{y}$ , we may write the Galerkin projection  $u^L(\mathbf{y}) \in \mathcal{X}^{h_L}$  of  $u(\mathbf{y})$  as

$$u^L(\mathbf{y}) = \sum_{l=1}^L u^l(\mathbf{y}) - u^{l-1}(\mathbf{y}), \quad (30)$$

where we define  $u^0(\mathbf{y}) \equiv 0$  (note that we will equivalently parametrize the approximations and spaces by  $l$  or  $h_l$ ). The idea behind our MLCSPG approach is to estimate every difference between consecutive levels of approximation (the *details*) via a single level CSPG as presented above. For the remaining, we let

$$\Delta u^l(\mathbf{y}) := u^l(\mathbf{y}) - u^{l-1}(\mathbf{y}), \quad 1 \leq l \leq L, \quad (31)$$

denote the difference between two scales of approximation.

As already outlined in the introduction, our method produces pointwise numerical approximations  $\widehat{\Delta u^l}(\mathbf{y})$  of  $\Delta u^l(\mathbf{y})$  via a (single level) CSPG method. For each level  $l$ , we choose a number  $m_l$  of parameter vectors  $\mathbf{y}_l^{(1)}, \dots, \mathbf{y}_l^{(m_l)}$ , compute the PG approximations  $u^l(\mathbf{y}_l^{(i)})$  and  $u^{l-1}(\mathbf{y}_l^{(i)})$  by solving the corresponding finite dimensional linear systems, and form the samples  $\Delta u^l(\mathbf{y}_l^{(i)}) = u^l(\mathbf{y}_l^{(i)}) - u^{l-1}(\mathbf{y}_l^{(i)})$ ,  $i = 1, \dots, m_l$ . From these samples, one approximates the coefficients in the tensorized Chebyshev expansion of  $\Delta u^l$  via weighted  $\ell_1$ -minimization (or any sparse recovery method). This yields approximations  $\widehat{\Delta u^l}(\mathbf{y})$ ,  $\ell = 1, \dots, L$ , and

$$u_{\text{MLCS}}^L(\mathbf{y}) := \sum_{l=1}^L \widehat{\Delta u^l}(\mathbf{y})$$

then provides an approximation of the targeted parametric solution  $u = u(\mathbf{y})$ . The convergence of our MLCSPG framework can be estimated via the triangle inequality,

$$\|u(\mathbf{y}) - u_{\text{MLCS}}^L(\mathbf{y})\|_{\mathcal{X}} \leq \|u(\mathbf{y}) - u^L(\mathbf{y})\|_{\mathcal{X}} + \sum_{l=1}^L \left\| \Delta u^l(\mathbf{y}) - \widehat{\Delta u^l}(\mathbf{y}) \right\|_{\mathcal{X}}. \quad (32)$$

For simplicity, we constrain our considerations to a functional  $\mathcal{G} \in \mathcal{X}'$  applied to the solution, leading to the real-valued QoI  $F(\mathbf{y}) = \mathcal{G}(u(\mathbf{y}))$  to be approximated. The above considerations apply verbatim when replacing  $u(\mathbf{y})$  by  $F(\mathbf{y})$ , and  $\Delta u^l$  by  $\Delta F^l$ , the levelwise PG approximation, and  $\widehat{\Delta u^l}(\mathbf{y})$  by  $\widehat{\Delta F^l}(\mathbf{y})$ . The triangle inequality leads to the error estimate

$$|F(\mathbf{y}) - F_{\text{MLCS}}^L(\mathbf{y})| \leq |F(\mathbf{y}) - F^L(\mathbf{y})| + \sum_{l=1}^L \left| \Delta F^l(\mathbf{y}) - \widehat{\Delta F^l}(\mathbf{y}) \right|. \quad (33)$$

The first term on the right hand side of Eq. (32) can be estimated using the uniform parametric regularity (16), the uniform parametric inf-sup condition (8) and the approximation property (17): for a regularity parameter  $0 < t \leq \bar{t}$  of the data  $f$ ,

$$\|u(\mathbf{y}) - u^L(\mathbf{y})\|_{\mathcal{X}} \leq C_t h_L^t \|f\|_{\mathcal{Y}_t'}.$$

Passing to the functional  $\mathcal{G} \in \mathcal{X}'_t$ , we obtain a superconvergence error bound for the Petrov-Galerkin-Finite Element Method (PG-FEM) via a classical Aubin-Nitsche duality argument [43]:

$$|F(\mathbf{y}) - F^L(\mathbf{y})| \leq C_{t+t'} h_L^{t+t'} \|f\|_{\mathcal{Y}_t'} \|\mathcal{G}\|_{\mathcal{X}'_t'}.$$

Our goal is to verify that the single-level result, Theorem 2, applies to all levels  $l = 1, \dots, L$ , and to obtain error bounds similar to the one in Eq. (20). We consider the Chebyshev expansions of the differences,

$$\mathfrak{d}u^l(\mathbf{y}) = \sum_{\nu \in \mathcal{F}} \mathfrak{d}u_{\nu}^l T_{\nu}(\mathbf{y}), \quad (34)$$

$$\Delta F^l(\mathbf{y}) = \sum_{\nu \in \mathcal{F}} \Delta F_{\nu}^l T_{\nu}(\mathbf{y}). \quad (35)$$

Assuming summability of the expansion in  $\ell_{\omega,p}(\mathcal{F})$ , we can apply Theorem 2 with a number of samples  $m_l \gtrsim s_l \log(s_l)^3 \log(N_l)$ , for suitable choices of  $s_1, \dots, s_L$ , and in particular we can use the error estimate (29) in terms of the (weighted) sparsity  $s_l$  for each level of approximation. This results in the bound

$$|\Delta F^l(\mathbf{y}) - \widehat{\Delta F^l}(\mathbf{y})| \leq C \left\| (\Delta F_{\nu}^l)_{\nu} \right\|_{\omega,p} s_l^{1-1/p}, \quad \text{for all } 1 \leq l \leq L, \quad (36)$$

where  $C > 0$  is a universal constant (independent of  $s_l, \mathbf{y}, l$ ). Theorem 2 applies only to finite orthonormal systems. Thus, for each  $l = 1, 2, \dots, L$ , the countably infinite index set  $\mathcal{F}$  has to be truncated to a finite, but possibly large, subset  $\Gamma_l$  of  $N_l := |\Gamma_l| < \infty$  many indices of the relevant (few) essential Chebyshev

coefficients in the parametric solution's gpc expansion. We describe a strategy for selecting the index sets  $\Gamma_l$  depending on  $s_l$  in Section 5.1. A good choice for the  $s_l$  turns out to be  $s_l \asymp 2^{(L-l)(t+t')p/(1-p)}$ , as will be derived ahead.

Finally, summing up the contributions from all discretization levels and drawing

$$m_l \gtrsim 2^{(L-l)(t+t')p/(1-p)} (L-l)^3 \log(N_l)$$

sample points per level will imply the error bounds in Theorem 9. The choice of this number of sampling points is justified in Section 4.3 and by the following result, whose proof is the purpose of the next section.

**Theorem 3.** *Let  $\{A(\mathbf{y}) : \mathbf{y} \in U\}$  be a parametric family of operators as defined in (5). Assume that the operator  $A_0$  is inf – sup stable. For  $B_j := A_0^{-1} A_j$  and for  $0 \leq t \leq \bar{t}$ , introduce the sequence*

$$\mathbf{b}_t := (b_{t,j})_{j \geq 1} \quad \text{with} \quad b_{t,j} := \|B_j\|_{\mathcal{L}(\mathcal{X}_t)} = \|B_j^*\|_{\mathcal{L}(\mathcal{Y}_t)}. \quad (37)$$

Let  $\mathbf{v} := (v_j)_{j \geq 1}$  be a sequence of weights with  $v_j \geq 1$  such that, for some  $p < 1$ ,

$$\sum_{j \geq 1} b_{t,j} v_j^{(2-p)/p} \leq \kappa_{\mathbf{v},p} < 1, \quad \text{and} \quad (38)$$

$$\sum_{j \geq 1} b_{t,j}^p v_j^{2-p} < \infty. \quad (39)$$

Let  $\rho$  be a  $\mathbf{b}_0 - \delta$ -admissible sequence of polyradia, with  $\delta = (1 - \kappa_{\mathbf{v},p})/2$ , i.e., such that

$$\sum_{j \geq 1} (\rho_j - 1) b_{0,j} \leq \delta. \quad (40)$$

Then the family of operators  $A(\mathbf{y})$  is uniformly inf – sup stable. Assume in addition that  $A_j \in \mathcal{L}(\mathcal{X}_t, \mathcal{Y}'_t)$ ,  $j \geq 0$ , are defined on the scale of smoothness spaces  $\mathcal{X}_t$  and that the approximation property (17) holds. Assume moreover that  $A_0 : \mathcal{X}_t \rightarrow \mathcal{Y}'_t$  is boundedly invertible and that the sequence  $\mathbf{b}_t$  is small, and that the polyradius  $\rho$  is  $\mathbf{b}_t - \delta_t$ -admissible, i.e.

$$\sum_{j \geq 1} b_{t,j} \leq \kappa_t < 1, \quad (41)$$

$$\sum_{j \geq 1} (\rho_j - 1) b_{t,j} \leq \delta_t, \quad (42)$$

for  $\delta_t < 1 - \kappa_t$ .

Then the affine-parametric family of operators  $\{A(\mathbf{y}) : \mathbf{y} \in U\}$  is uniformly boundedly invertible in  $\mathcal{L}(\mathcal{X}_t, \mathcal{Y}'_t)$ , and there hold bounds on the Chebyshev gpc coefficients

$$\|\Delta u_\nu^l\|_{\mathcal{X}} \leq C h_t^t \|f\|_{\mathcal{Y}'_t} \rho^{-\nu}, \quad \text{and} \quad |\Delta F_\nu^l| \leq C h_t^{t+t'} \|\mathcal{G}\|_{\mathcal{X}'_t} \|f\|_{\mathcal{Y}'_t} \rho^{-\nu} \quad \text{for all } \nu \in \mathcal{F}. \quad (43)$$

Moreover, for each  $\nu \in \mathcal{F}$ , there exists a  $\delta$ -admissible sequence  $\rho = \rho(\nu)$  satisfying (42) such that the sequence with components  $\rho(\nu)^{-\nu} = \prod_{j \geq 1} \rho(\nu)_j^{-\nu_j}$ ,  $\nu \in \mathcal{F}$ , satisfies  $(\rho(\nu)^{-\nu})_{\nu \in \mathcal{F}} \in \ell_{\omega,p}$ , where

$$\omega_\nu := \theta^{\|\nu\|_0} \mathbf{v}^\nu = \theta^{\|\nu\|_0} \prod_{j \geq 1} v_j^{\nu_j}, \quad \nu \in \mathcal{F}. \quad (44)$$

We want to stress once again that the result presented above is written without the explicit dependence of the weight sequence  $\mathbf{v}$  on the regularity parameter  $t$ . Moreover, we note that the conditions (37) - (42) are, for  $t > 0$ , strictly stronger than the summability conditions which were required in the single-level PG analysis in [48].

**4.2. Summability of the Chebyshev expansions.** This section provides the proof of the core result of the present paper, Theorem 3. We show that under general assumptions, the parametric solution's sequence of Chebyshev coefficients  $(\Delta F_\nu^l)_{\nu \in \mathcal{F}} \in \ell_{p,\omega}$ , and in particular that the following a priori estimate holds:

$$\left\| (\Delta F_\nu^l)_{\nu \in \mathcal{F}} \right\|_{\omega,p} \leq C h_t^{t+t'} \|f\|_{\mathcal{Y}'_t} \|\mathcal{G}\|_{\mathcal{X}'_t} \|(\rho(\nu)^{-\nu})_{\nu \in \mathcal{F}}\|_{\omega,p}. \quad (45)$$

The main novel point of this estimate is the scaling of the right hand side with  $h_l^{t+t'}$ . The proof of this assertion is structured in three main steps, analogously to [21, 48]. First we show that the difference between levels is holomorphic in polydiscs. Then, this holomorphy is used to bound the norm of any Chebyshev coefficient. This norm depends on a sequence of radii of holomorphy  $\rho$ . Finally, we construct a sequence of radii and weights such that the sequence of coefficients is  $\ell_{\omega,p}$  summable.

**4.2.1. Holomorphy.** This first part shows that, under some uniform invertibility assumption of the family of (complexified) operators  $A(\mathbf{z})$  (which are satisfied in particular for the affine-parametric family considered here), the solutions are holomorphic with respect to any finite set of variables. This then allows to use Cauchy's integral formula to estimate the norm of the Chebyshev coefficients.

**Theorem 4.** *For some  $\mathcal{O} \subset \mathbb{C}^{\mathbb{N}}$  with  $\mathcal{O} \supset U$ , assume that the complex inf – sup conditions (13) hold with constant  $\mu_{\mathbb{C}}$  uniformly for  $\mathbf{z} \in \mathcal{O}$ . If the solution map  $\mathcal{O} \ni \mathbf{z} \rightarrow u(\mathbf{z}) \in \mathcal{X}$  is holomorphic with respect to any finite set of parameters, then*

- (1) *for any level  $l$  of PG discretization (corresponding to the discretization parameter  $h_l = 2^{-l}h_0$  for a given  $h_0 > 0$  sufficiently small), the parametric Galerkin projections  $\mathcal{O} \ni \mathbf{z} \rightarrow u^l(\mathbf{z}) \in \mathcal{X}^l$  are holomorphic with respect to any finite subset of the sequence  $\mathbf{z} \in \mathcal{O}$ , with domains of holomorphy whose size is independent of  $l$ , i.e. of the discretization parameter  $h_l$ ,*
- (2) *the Petrov-Galerkin projections are quasi-optimal, uniformly with respect to the level of approximation  $l$  and the vector of (complex) parameters  $\mathbf{z} \in \mathcal{O}$ :*

$$\|u(\mathbf{z}) - u^l(\mathbf{z})\|_{\mathcal{X}} \leq \left(1 + \frac{C}{\mu_{\mathbb{C}}}\right) \inf_{v^l \in \mathcal{X}^l} \|u(\mathbf{z}) - v^l\|_{\mathcal{X}}.$$

*Proof.* The holomorphy follows from the linearity of the PG approximation as stated in Proposition 1. The quasi optimality is obtained in the same way as in the real case.  $\square$

The next corollary which uses the notation (31) follows directly.

**Corollary 1.** *Under the conditions above, if in addition the approximation property of the discretization spaces holds for complex parameters  $\mathbf{z} \in \mathcal{O}$ , then for any two consecutive discretization levels  $l$  and  $l+1$ ,  $l \geq 0$ , the mappings  $\mathcal{O} \ni \mathbf{z} \mapsto \Delta u^l(\mathbf{z}) \in \mathcal{X}$  are holomorphic with respect to any finite set of variables and satisfy the uniform bound*

$$\sup_{\mathbf{z} \in \mathcal{O}} \|\Delta u^l(\mathbf{z})\|_{\mathcal{X}} \leq C'_{t,\mu_{\mathbb{C}}} h_l^t \sup_{\mathbf{z} \in \mathcal{O}} \|u(\mathbf{z})\|_{\mathcal{X}_t}.$$

*Proof.* The statement is a consequence of the previous results and the triangle inequality:

$$\begin{aligned} \sup_{\mathbf{z} \in \mathcal{O}} \|\Delta u^l(\mathbf{z})\|_{\mathcal{X}} &\leq \left(1 + \frac{C}{\mu_{\mathbb{C}}}\right) \sup_{\mathbf{z} \in \mathcal{O}} \left( \inf_{v^l \in \mathcal{X}^l} \|u(\mathbf{z}) - v^l\|_{\mathcal{X}} + \inf_{v^{l-1} \in \mathcal{X}^{l-1}} \|u(\mathbf{z}) - v^{l-1}\|_{\mathcal{X}} \right) \\ &\leq \left(1 + \frac{C}{\mu_{\mathbb{C}}}\right) \sup_{\mathbf{z} \in \mathcal{O}} C_t (h_l^t \|u(\mathbf{z})\|_{\mathcal{X}_t} + h_{l-1}^t \|u(\mathbf{z})\|_{\mathcal{X}_t}) = C'_{t,\mu_{\mathbb{C}}} h_l^t \sup_{\mathbf{z} \in \mathcal{O}} \|u(\mathbf{z})\|_{\mathcal{X}_t}. \end{aligned}$$

$\square$

**4.2.2. Nominal inf-sup conditions imply uniform inf-sup conditions.** The preceding result, Theorem 4, requires the validity of a *uniform discrete inf-sup condition* for the PG discretization; here, uniformity is understood with respect to the discretization parameter  $h > 0$  and with respect to the parameter sequence  $\mathbf{z} \in \mathcal{O}$  in Theorem 4 or with respect to  $\mathbf{y} \in U$  in (8), respectively. In what follows, we assume that the two one-parameter families of dense subspaces  $\{\mathcal{X}^h\}_{h>0} \subset \mathcal{X}$  and  $\{\mathcal{Y}^h\}_{h>0} \subset \mathcal{Y}$  are of equal, finite dimension  $N^h = \dim(\mathcal{X}^h) = \dim(\mathcal{Y}^h)$  and are stable for the nominal operator  $A_0 \in \mathcal{L}(\mathcal{X}, \mathcal{Y}')$  in (5), i.e., the discrete inf-sup conditions hold

$$\inf_{0 \neq w^h \in \mathcal{X}^h} \sup_{0 \neq v^h \in \mathcal{Y}^h} \frac{\langle A_0 w^h, v^h \rangle}{\|w^h\|_{\mathcal{X}} \|v^h\|_{\mathcal{Y}}} \geq \mu_0 > 0, \quad \inf_{0 \neq v^h \in \mathcal{Y}^h} \sup_{0 \neq w^h \in \mathcal{X}^h} \frac{\langle A_0 w^h, v^h \rangle}{\|w^h\|_{\mathcal{X}} \|v^h\|_{\mathcal{Y}}} \geq \mu_0 > 0. \quad (46)$$

**Theorem 5.** *Suppose that the parametric operators  $A(\mathbf{y})$ ,  $A(\mathbf{z})$  are affine-parametric, as in (5). Assume further that for  $t \geq 0$  the sequences  $\mathbf{b}_t = (b_{t,j})_{j \geq 1}$  in (37) are small, in the sense that (41) holds. Then, (41) with  $t = 0$  implies that the discrete inf-sup conditions (8) hold uniformly with respect to  $\mathbf{y} \in U$ .*

Moreover, if the sequence of polyradii  $\rho = (\rho_j)_{j \geq 1}$  is admissible, in the sense that (40) holds for  $t = 0$  and for some  $\delta < 1 - \kappa_0$ , then the complex-parametric  $A(\mathbf{z})$  1) satisfies the uniform inf-sup conditions (13) for  $\mathbf{z} \in \mathcal{D}_\rho = \bigotimes_{j \geq 1} \mathcal{D}_{\rho_j}$ , where  $\mathcal{D}_{\rho_j} := \{z \in \mathbb{C} : |z| \leq \rho_j\}$ , and 2) is holomorphic with respect to any finite set of variables in  $\mathcal{D}_\rho$ .

Similarly,  $A(\mathbf{z})$  is invertible in  $\mathcal{L}(\mathcal{X}_t, \mathcal{Y}'_t)$  uniformly for  $\mathbf{z} \in \mathcal{D}_\rho$  if  $\rho$  is  $\delta_t$ -admissible w.r.t. the sequence  $\mathbf{b}_t = (b_{t,j})_{j \geq 1}$  with  $\delta_t < 1 - \kappa_t$ , where  $b_{t,j} := \|A_0^{-1} A_j\|_{\mathcal{L}(\mathcal{X}_t)}$ .

*Proof.* Let  $\mathbf{b}_0$  be such that condition (41) holds with  $t = 0$ . Since  $A_0$  is assumed to be boundedly invertible, we can write  $A(\mathbf{y}) = A_0 \left( I + \sum_{j \geq 1} y_j A_0^{-1} A_j \right)$  and estimate

$$\left\| \sum_{j \geq 1} y_j A_0^{-1} A_j \right\|_{\mathcal{L}(\mathcal{X})} \leq \sum_{j \geq 1} |y_j| b_{0,j} \leq \sum_{j \geq 1} b_{0,j} := \kappa_0 < 1.$$

It follows from a perturbation (Neumann series) argument that the operator  $A(\mathbf{y})$  is uniformly boundedly invertible. The discrete inf – sup conditions hold with  $\mu \leq \mu_0(1 - \kappa_0)$ .

One may extend this argument to the complexified operator  $A(\mathbf{z})$  defined for  $\mathbf{z} \in \mathcal{D}_\rho$ . This yields

$$\left\| \sum_{j \geq 1} z_j A_0^{-1} A_j \right\|_{\mathcal{L}(\mathcal{X})} \leq \sum_{j \geq 1} |z_j| b_{0,j} \leq \sum_{j \geq 1} \rho_j b_{0,j} := \delta + \kappa_0 < 1.$$

Therefore, the complex inf – sup conditions (13) hold with constant  $\mu_{\mathbb{C}} \leq \mu_0(\delta + \kappa)$ .

The proof of the uniform invertibility in  $\mathcal{L}(\mathcal{X}_t, \mathcal{Y}'_t)$  follows in a similar fashion.

The operator  $A(\mathbf{z})$  being invertible, we may write, for  $\mathbf{z} \in \mathcal{D}_\rho$  and some  $k \in \mathbb{N}$ ,

$$u(\mathbf{z}) = \left( A_0 + \sum_{j \neq k} A_0 z_j A_0^{-1} A_j + A_0 z_k A_0^{-1} A_k \right)^{-1} f = \left( I + \sum_{j \neq k} z_j B_j + z_k B_k \right)^{-1} A_0^{-1} f.$$

Whence,  $u(\mathbf{z})$  is holomorphic with respect to  $z_k \in \mathcal{D}_{\rho_k}$  as the image of  $f$  via a resolvent operator. Hartogs' theorem concludes the holomorphy with respect to any finite set of parameters.  $\square$

**4.2.3. Norm bounds on the Chebyshev gpc coefficients.** We now estimate the magnitudes of the Chebyshev coefficients. These estimates are used in the next section to show the  $\ell_{\omega,p}$  summability of the sequence of Chebyshev coefficients. We recall that  $\mathcal{E}_\rho = \bigotimes_{j \geq 1} \mathcal{E}_{\rho_j}$  is a product of Bernstein ellipses  $\mathcal{E}_{\rho_j} = \{(z + z^{-1})/2, z \in \mathbb{C} : |z| = \rho_j\}$  and let  $E_\rho = \bigotimes_{j \geq 1} E_{\rho_j}$  be the product of the open regions  $E_{\rho_j} := \{(z + z^{-1})/2, z \in \mathbb{C} : 1 \leq |z| < \rho_j\}$  bounded by the Bernstein ellipses  $\mathcal{E}_{\rho_j}$ . We note that  $E_\rho$  and  $\mathcal{E}_\rho$  are contained in  $\mathcal{D}_\rho$  so that in particular under the assumptions of Theorem 5 we are in the setting of the next result.

**Theorem 6.** *Let  $\nu \in \mathcal{F}$ . Assume that the discretization spaces have the approximation property (17). Additionally, assume that there exists a sequence  $\rho = (\rho_j)_{j \geq 1}$ , with  $\rho_j > 1$  such that the complex extension  $\mathbf{z} \mapsto \Delta u^l(\mathbf{z})$  is holomorphic with respect to any finite set of variables on  $E_\rho$  and with  $A(\mathbf{z}) \in \mathcal{L}(\mathcal{X}_t, \mathcal{Y}'_t)$  being uniformly boundedly invertible for every  $\mathbf{z} \in \mathcal{E}_\rho$ . Then the Chebyshev coefficients of the difference  $\Delta u^l = u^l - u^{l-1}$  can be estimated as*

$$\|\Delta u^l_\nu\|_{\mathcal{X}} \leq Ch_t^{t'} \|f\|_{\mathcal{Y}'_t} \rho^{-\nu}.$$

If in addition we assume smoothness for the functional, i.e.  $\mathcal{G} \in \mathcal{X}'_t$ , for some  $0 < t' \leq \bar{t}$ , then it holds

$$|\Delta F^l_\nu| \leq Ch_t^{t+t'} \|f\|_{\mathcal{Y}'_t} \|\mathcal{G}\|_{\mathcal{X}'_t} \rho^{-\nu}, \quad (47)$$

where the constants depend on the smoothness parameters  $t$  and  $t'$  but not on  $h_l$ .

*Proof.* The proof is similar to the one in [48] with appropriate modifications due to the introduction of the levels. The tensorized Chebyshev polynomials being orthogonal, it holds

$$\Delta u^l_\nu = \int_U \Delta u^l(\mathbf{y}) T_\nu(\mathbf{y}) d\eta(\mathbf{y}).$$

Consider the multi-index  $\nu = n\mathbf{e}_1 = (n, 0, 0 \dots) \in \mathcal{F}$  and split the parameter space as  $U = [-1, 1] \times U'$ , then any parameter sequence  $\mathbf{y}$  can be written as  $\mathbf{y} = (y_1, \mathbf{y}')$  with  $y_1 \in [-1, 1]$ . Thus

$$\Delta u_{n\mathbf{e}_1}^l = \int_{U'} \int_{-1}^{+1} T_n(t) \Delta u^l(t, \mathbf{y}') \frac{dt}{\pi\sqrt{1-t^2}} d\eta(\mathbf{y}'). \quad (48)$$

With the change of variables  $t = \cos(\phi)$  we obtain

$$\int_{-1}^{+1} T_n(t) \Delta u^l(t, \mathbf{y}') \frac{dt}{\pi\sqrt{1-t^2}} = \frac{\sqrt{2}}{\pi} \int_0^\pi \cos(n\phi) \Delta u^l(\cos(\phi), \mathbf{y}') d\phi = \frac{1}{\sqrt{2}\pi} \int_{-\pi}^{+\pi} \cos(n\phi) \Delta u^l(\cos(\phi), \mathbf{y}') d\phi.$$

This gives

$$\begin{aligned} \int_{-1}^{+1} T_n(t) \Delta u^l(t, \mathbf{y}') \frac{dt}{\pi\sqrt{1-t^2}} &= \frac{1}{\sqrt{2}\pi i} \int_{|z|=1} \frac{z^n + z^{-n}}{2} \Delta u^l\left(\frac{z+z^{-1}}{2}, \mathbf{y}'\right) \frac{dz}{z} \\ &= \frac{1}{2\sqrt{2}i\pi} \int_{|z|=1} z^{n-1} \Delta u^l\left(\frac{z+z^{-1}}{2}, \mathbf{y}'\right) dz + \frac{1}{2\sqrt{2}i\pi} \int_{|z|=1} z^{-n-1} \Delta u^l\left(\frac{z+z^{-1}}{2}, \mathbf{y}'\right) dz. \end{aligned}$$

Due to the assumption that the extension  $z \rightarrow \Delta u^l(z)$  to  $E_\rho$  is holomorphic, the mappings

$$z \mapsto z^{n-1} \Delta u^l\left(\frac{z+z^{-1}}{2}, \mathbf{y}'\right), \quad \text{and} \quad z \mapsto z^{-n-1} \Delta u^l\left(\frac{z+z^{-1}}{2}, \mathbf{y}'\right)$$

are analytic on  $E_{\rho_1}$ . By Cauchy's theorem it follows, for  $1 < \sigma < \rho_1$ , that

$$\begin{aligned} \int_{-1}^{+1} T_n(t) \Delta u^l(t, \mathbf{y}') \frac{dt}{\pi\sqrt{1-t^2}} &= \frac{1}{2\sqrt{2}i\pi} \int_{|z|=\sigma^{-1}} z^{n-1} \Delta u^l\left(\frac{z+z^{-1}}{2}, \mathbf{y}'\right) dz \\ &\quad + \frac{1}{2\sqrt{2}i\pi} \int_{|z|=\sigma} z^{-n-1} \Delta u^l\left(\frac{z+z^{-1}}{2}, \mathbf{y}'\right) dz. \end{aligned}$$

Now notice that  $z \mapsto \Delta u^l(z, \mathbf{y}')$  is bounded by  $C'h_i^t \|f\|_{\mathcal{Y}_i}$  (in  $\mathcal{X}$ ) in a polydisc containing in  $E_\rho$ . Indeed, the approximation property of the discretization spaces, see Corollary 1, together with the bounded invertibility in the smoothness spaces, ensures

$$\sup_{z \in \mathcal{E}_\rho} \|\Delta u^l(z)\|_{\mathcal{X}} = \sup_{z \in \mathcal{E}_\rho} \|u^l(z) - u^{l-1}(z)\|_{\mathcal{X}} \leq C'h_i^t \sup_{z \in \mathcal{E}_\rho} \|u(z)\|_{\mathcal{X}_i} \leq C'h_i^t \|f\|_{\mathcal{Y}_i}. \quad (49)$$

It follows that

$$\begin{aligned} \left\| \int_{-1}^{+1} T_n(t) \Delta u^l(t, \mathbf{y}') \frac{dt}{\pi\sqrt{1-t^2}} \right\|_{\mathcal{X}} &\leq \frac{1}{2\sqrt{2}\pi} \int_{|z|=\sigma^{-1}} |z^{n-1}| \left\| \Delta u^l\left(\frac{z+z^{-1}}{2}, \mathbf{y}'\right) \right\|_{\mathcal{X}} dz \\ &\quad + \frac{1}{2\sqrt{2}\pi} \int_{|z|=\sigma} |z^{-n-1}| \left\| \Delta u^l\left(\frac{z+z^{-1}}{2}, \mathbf{y}'\right) \right\|_{\mathcal{X}} dz \\ &\leq \frac{1}{2\sqrt{2}\pi\sigma^{n-1}} 2\pi C'h_i^t \|f\|_{\mathcal{Y}_i} \sigma^{-1} + \frac{1}{2\sqrt{2}\pi\sigma^{n+1}} 2\pi\sigma C'h_i^t \|f\|_{\mathcal{Y}_i} \\ &= \sqrt{2} C'h_i^t \|f\|_{\mathcal{Y}_i} \sigma^{-n}. \end{aligned} \quad (50)$$

This bound is valid for any  $\sigma < \rho_1$  and hence holds up to  $\sigma = \rho_1$ .

Finally, inserting Eq. (50) back into Eq. (48) after integrating over  $\mathbf{y}' \in U'$  with respect to the probability measure  $d\eta(\mathbf{y}')$  yields

$$\|\Delta u_{n\mathbf{e}_1}^l\|_{\mathcal{X}} \leq C'h_i^t \|f\|_{\mathcal{Y}_i} \rho^{-n}.$$

Similarly, given any  $\nu \in \mathcal{F}$ , it follows that

$$\|\Delta u_\nu^l\|_{\mathcal{X}} \leq C' h_t^t \|f\|_{\mathcal{Y}'_t} \rho^{-\nu},$$

by applying Cauchy's integral formula in  $\mathbb{C}$  with respect to each variable  $z_j$  for  $j \in \{j : \nu_j \neq 0\}$ .

The Chebyshev coefficients of the functional are estimated in a similar manner, using (20),

$$|\Delta F_\nu^l| \leq C' h_t^{t+t'} \|f\|_{\mathcal{Y}'_t} \|\mathcal{G}\|_{\mathcal{X}'_t} \rho^{-\nu}. \quad (51)$$

□

**4.2.4. Summability of the sequence of Chebyshev gpc coefficients.** It remains to prove the existence of a  $\delta$ -admissible polyradial  $\rho$  (depending on  $\nu$ ) and to verify the  $\ell_{\omega,p}$ -summability of the right hand side of (47) with respect to  $\nu \in \mathcal{F}$ , i.e., of the sequence  $(\rho(\nu)^{-\nu})_{\nu \in \mathcal{F}}$ . Hereby, we identify suitable weights  $\omega = (\omega_\nu)_{\nu \in \mathcal{F}}$  as well. In contrast to unweighted  $\ell_p$ -summability [20, 21], weighted  $\ell_{\omega,p}$ -summability – considered first in [48] – requires stronger assumptions on the sequence  $(b_{0,j})_{j \in \mathbb{N}}$  used as base for the  $\delta$ -admissibility (40). Namely, with  $\mathbf{v} = (v_j)_{j \in \mathbb{N}}$  and  $v_j \geq 1$ , we ask for properties (38) and (39) to be valid.

**Theorem 7.** *Let  $\mathbf{v}$  be a sequence of weights fulfilling the summability conditions (38) and (39) and let  $\omega_\nu := \theta^{\|\nu\|_0} \mathbf{v}^\nu$  for any  $\nu \in \mathcal{F}$  and some  $\theta \geq 1$ . There exists a sequence of polyradial  $(\rho(\nu))_{\nu \in \mathcal{F}}$  such that*

- i) for each  $\nu \in \mathcal{F}$ ,  $\rho = \rho(\nu)$  is  $\delta$ -admissible, with  $\delta = (1 - \kappa_{\mathbf{v},p})/2$ , and
- ii)  $\|(\rho(\nu)^{-\nu})_{\nu \in \mathcal{F}}\|_{\omega,p} \leq K_{\theta,p} < \infty$ .

*Proof.* Full details of the argument can be found in [48]; here, we only indicate the main steps, in particular the construction of a sequence of weights  $\omega$  and an associated, admissible sequence of polyradial.

For the weights  $\mathbf{v}$  and a constant  $\theta \geq 1$ , we define the sequence of weights

$$\omega_\nu(\theta) := \theta^{\|\nu\|_0} \mathbf{v}^\nu = \theta^{\|\nu\|_0} \prod_{j:\nu_j \neq 0} v_j^{\nu_j}, \quad \text{for all } \nu \in \mathcal{F}. \quad (52)$$

Because of (38), there exists a finite set  $E \subset \mathbb{N}$  such that, with  $F := \mathbb{N} \setminus E$ ,

$$\sum_{j \in F} v_j^{(2-p)/p} b_{0,j} \leq \frac{\delta}{8\theta^{(2-p)/p}}.$$

For a given constant  $\alpha > 1$  with  $(\alpha - 1) \sum_{j \in E} v_j^{(2-p)/p} b_{0,j} < \delta/2$ , we define the sequences of polyradial (generally depending on  $\nu$ ) as

$$\rho_j(\nu) = \begin{cases} \alpha v_j^{(2-p)/p}, & j \in E, \\ \max \left\{ v_j^{(2-p)/p}, \frac{\nu_j}{2|\nu_F| b_j} \right\}, & j \in F, \end{cases} \quad (53)$$

where we used the notation  $|\nu_F| := \sum_{j \in F} \nu_j$ . The  $\delta$ -admissibility of this sequence, as well as its  $\ell_{\omega,p}$  summability, ensuring the summability of the Chebyshev expansion of the differences, have been proved in [48, Theorem 4.2]. □

Combining the estimate (51) with the  $\ell_{\omega,p}$  summability of the sequence  $\rho$  yields

$$\|\Delta F^l\|_{\omega,p} \leq C h_t^{t+t'} \|f\|_{\mathcal{Y}'_t} \|\mathcal{G}\|_{\mathcal{X}'_t} \|(\rho^{-\nu})_{\nu \in \mathcal{F}}\|_{\omega,p}. \quad (54)$$

Consequently, with Eq. (36) it follows

$$|\Delta F^l(\mathbf{y}) - \widehat{\Delta F^l}(\mathbf{y})| \leq C s_t^{1-1/p} h_t^{t+t'} \|f\|_{\mathcal{Y}'_t} \|\mathcal{G}\|_{\mathcal{X}'_t} \|(\rho(\nu)^{-\nu})_{\nu \in \mathcal{F}}\|_{\omega,p}. \quad (55)$$

Theorem 3 is a direct consequence of the results in this section. Indeed, the bounded invertibility in the smoothness spaces of  $A_0 \in \mathcal{L}(\mathcal{X}_t, \mathcal{Y}'_t)$  together with the summability (42) implies the uniform bounded invertibility of the operator  $A(\mathbf{z}) \in \mathcal{L}(\mathcal{X}_t, \mathcal{Y}'_t)$ , via a perturbation argument as stated in Theorem 5. This ensures the applicability of Theorem 6 (which itself depends on the two previous theorems). Theorem 7 finally proves the existence of both a positive weight sequence  $\omega$  and a sequence of polyradial  $\rho$  as well as the  $\ell_{\omega,p}$  summability.

**4.3. Rate of convergence of the MLCSPG method.** To simplify the exposition, we only derive the bounds for the approximation of a functional of the parametric solution. The results can be applied mutatis mutandis to derive the convergence rates for the full solution  $u(\mathbf{y})$ , – once the details of the (single level) compressive sensing scheme for the approximation of the full solution are worked out. We continue the estimate in (33) as follows:

$$\begin{aligned} |F(\mathbf{y}) - F_{\text{MLCS}}^L(\mathbf{y})| &\leq |F(\mathbf{y}) - F^L(\mathbf{y})| + \sum_{l=1}^L \left| \Delta F^l(\mathbf{y}) - \widehat{\Delta F^l}(\mathbf{y}) \right| \\ &\leq Ch_L^{t+t'} \|f\|_{\mathcal{Y}'_i} \|\mathcal{G}\|_{\mathcal{X}'_{i'}} + \sum_{l=1}^L Cs_l^{1-1/p} \|\Delta F^l\|_{\omega,p} \\ &\leq C \|f\|_{\mathcal{Y}'_i} \|\mathcal{G}\|_{\mathcal{X}'_{i'}} \left( h_L^{t+t'} + \sum_{l=1}^L s_l^{1-1/p} h_l^{t+t'} \|(\rho(\nu)^{-\nu})_{\nu \in \mathcal{F}}\|_{\omega,p} \right). \end{aligned}$$

We absorb the norm  $\|(\rho(\nu)^{-\nu})_{\nu \in \mathcal{F}}\|_{\omega,p}$  into the constant  $C > 0$ , yielding

$$|F(\mathbf{y}) - F_{\text{MLCS}}^L(\mathbf{y})| \leq C \|f\|_{\mathcal{Y}'_i} \|\mathcal{G}\|_{\mathcal{X}'_{i'}} \left( h_L^{t+t'} + \sum_{l=1}^L s_l^{1-1/p} h_l^{t+t'} \right).$$

Using that the levels are related via  $h_l = h_{l-1}/2$  we obtain

$$|F(\mathbf{y}) - F_{\text{MLCS}}^L(\mathbf{y})| \leq C \|f\|_{\mathcal{Y}'_i} \|\mathcal{G}\|_{\mathcal{X}'_{i'}} h_L^{t+t'} \left( 1 + \sum_{l=1}^L s_l^{1-1/p} 2^{(L-l)(t+t')} \right).$$

We balance sampling and discretization errors on each mesh level  $l$  in this bound. Thus the choice

$$s_l \asymp 2^{(L-l)(t+t')p/(1-p)} = 2^{(L-l)\sigma_p(t+t')}, \quad \text{with } \sigma_p(t) = \frac{tp}{1-p}, \quad (56)$$

implies an overall error bound of

$$|F(\mathbf{y}) - F_{\text{MLCS}}^L(\mathbf{y})| \leq C \|f\|_{\mathcal{Y}'_i} \|\mathcal{G}\|_{\mathcal{X}'_{i'}} (L+1) h_L^{t+t'} = C \|f\|_{\mathcal{Y}'_i} \|\mathcal{G}\|_{\mathcal{X}'_{i'}} (|\log(h_L)| + 1) h_L^{t+t'}.$$

From the choice of the sparsities (56) together with (27), it follows that a number of samples per level scaling as

$$m_l \asymp s_l \max\{\log^3(s_l) \log(N_l), \log(1/\gamma)\} \asymp 2^{(L-l)\sigma_p(t+t')} \max\{(L-l)^3 \log(N_l), \log(1/\gamma)\} \quad (57)$$

is sufficient for the error bound (55) to be valid at level  $l$  with probability exceeding  $1 - \gamma$ . Note that the size  $N_l$  of the initial index set  $\Gamma_l$  may depend on  $s_l$  and on the choice of weights  $\omega$ . More details are given in the next section. The global error in  $L^2$  is bounded as in Eq. (33),

$$\|F - F_{\text{MLCS}}^L\|_2 \leq \|F - F^L\|_2 + \sum_{l=1}^L \|\Delta F^l - \widehat{\Delta F^l}\|_2. \quad (58)$$

The first term is computed using the uniform bound (20) and the fact that  $\eta$  is a probability measure. To compute the sum, it suffices to apply the  $\ell_2$ -error bound in (29) to the  $L$  details,  $\|\Delta F^l - \widehat{\Delta F^l}\|_2 \leq Ds_l^{1/2-1/p} \|\Delta F^l\|_{\omega,p}$ . Hence, applying (20) to the first term and combining the  $\ell_2$  bound in (29) with the prior estimate (45) and the number of samples (56) in the terms in the sum yields

$$\|F - F_{\text{MLCS}}^L\|_2 \leq C \|f\|_{\mathcal{Y}'_i} \|\mathcal{G}\|_{\mathcal{X}'_{i'}} h_L^{t+t'} \left( 1 + \sum_{l=1}^L 2^{(t+t')(L-l)\frac{p-2}{2(1-p)}} 2^{(L-l)(t+t')} \right) = C' \|f\|_{\mathcal{Y}'_i} \|\mathcal{G}\|_{\mathcal{X}'_{i'}} h_L^{t+t'}.$$

Alternatively, one can also balance the number of samples with the discretization error to reach a prescribed  $L_2$  error of  $\mathcal{O}(h_L^{t+t'})$  by combining Eq. (58) with the compressed sensing approximation (29):

$$\begin{aligned} \|F - F_{\text{MLCS}}^L\|_2 &\leq Ch_L^{t+t'} \|f\|_{\mathcal{Y}'_t} \|\mathcal{G}\|_{\mathcal{X}'_{t'}} + C \sum_{l=1}^L s_l^{1/2-1/p} h_l^{t+t'} \|f\|_{\mathcal{Y}'_t} \|\mathcal{G}\|_{\mathcal{X}'_{t'}} \|(\rho(\nu)^{-\nu})_{\nu \in \mathcal{F}}\|_{\omega, p} \\ &\leq C \|f\|_{\mathcal{Y}'_t} \|\mathcal{G}\|_{\mathcal{X}'_{t'}} h_L^{t+t'} \left( 1 + \sum_{l=1}^L s_l^{1/2-1/p} 2^{(L-l)(t+t')} \right). \end{aligned}$$

In this case, choosing

$$s_l \asymp 2^{\frac{(L-l)(t+t')2p}{2-p}} \quad (59)$$

ensures the  $L_2$  error bound

$$\|F - F_{\text{MLCS}}^L\|_2 \leq Ch_L^{t+t'} (1 + |\log h_L|) \|f\|_{\mathcal{Y}'_t} \|\mathcal{G}\|_{\mathcal{X}'_{t'}}. \quad (60)$$

## 5. IMPLEMENTATION ASPECTS

This section describes several aspects that are relevant for the numerical applicability of the theoretical approach introduced above. In particular, we investigate the truncation of the (potentially infinite) sequence of parameters to a finite subset, and specify initial choices of finite index sets  $\Lambda \subset \mathcal{F}$  that are guaranteed to contain the support of the best (weighted)  $s$ -term approximation of the solution and can be used within weighted  $\ell_1$ -minimization or other CS algorithms.

**5.1. Dimension truncation.** So far, we have worked on a purely theoretical level, where the parameter vector is potentially infinite (but countable). To ensure the applicability of the results, we have to verify that truncating the parameter vector to a finite dimensional space (yet allowing this truncation to be rather large) still allows for reliable approximations.

We consider the weak solutions of the truncated version of Eq. (1):

$$\text{Find } u^{(B)} \in \mathcal{X}, \text{ such that } \langle A^{(B)}(\mathbf{y})u^{(B)}, v \rangle = \langle f, v \rangle \quad \text{for all } v \in \mathcal{Y}, \quad (61)$$

where the operator  $A^{(B)}(\mathbf{y})$  is defined, for a finite  $B \in \mathbb{N}$ , as  $A(y_1, y_2, \dots, y_B, 0, 0, \dots)$ .

In particular, we assume some decay of the *energy* of the operator  $A(\mathbf{y})$  (i.e. assuming a certain order on the parameters) such that for any  $\varepsilon > 0$ , there exists  $B := B(\varepsilon, A)$  with

$$\|A(\mathbf{y}) - A^{(B)}(\mathbf{y})\|_{\mathcal{L}(\mathcal{X}, \mathcal{Y}')} \leq \varepsilon \mu, \quad \forall \mathbf{y} \in U, \quad (62)$$

where  $\mu$  is the constant appearing in the inf – sup conditions (8).

In this case, the following generalization of results in [24] holds.

**Proposition 2.** *Assume the operator  $A$  satisfies the (continuous) inf – sup conditions (8) and the decay property (62). Then for any accuracy parameter  $\varepsilon$ , there exists a truncation parameter  $B \in \mathbb{N}$  such that the solutions to the truncated problem (61) and to the original problem (1) are close to each other in the following sense*

$$\|u^{(B)}(\mathbf{y}) - u(\mathbf{y})\|_{\mathcal{X}} \leq \frac{C\varepsilon}{\mu} \|f\|_{\mathcal{Y}'}, \quad (63)$$

where  $u^{(B)}(\mathbf{y})$  is the solution of the truncated problem (61).

*Proof.* The weak solutions are characterized by

$$\begin{aligned} \text{Find } u(\mathbf{y}) \text{ such that } \quad &\langle A(\mathbf{y})u(\mathbf{y}), v \rangle = \langle f, v \rangle \quad \text{for all } v \in \mathcal{Y}, \\ \text{Find } u^{(B)}(\mathbf{y}) \text{ such that } \quad &\langle A^{(B)}(\mathbf{y})u^{(B)}(\mathbf{y}), v \rangle = \langle f, v \rangle \quad \text{for all } v \in \mathcal{Y}. \end{aligned}$$

Since these equalities hold for all  $v$ , they imply the orthogonality conditions

$$\langle A(\mathbf{y})u(\mathbf{y}) - A^{(B)}(\mathbf{y})u^{(B)}(\mathbf{y}), v \rangle = 0 \quad \text{for all } v \in \mathcal{Y}.$$

Rearranging the terms yields  $\langle A(\mathbf{y})(u(\mathbf{y}) - u^{(B)}(\mathbf{y})), v \rangle = -\langle (A(\mathbf{y}) - A^{(B)}(\mathbf{y}))u^{(B)}(\mathbf{y}), v \rangle$  for all  $v \in \mathcal{Y}$ . This means that  $u(\mathbf{y}) - u^{(B)}(\mathbf{y})$  is the weak solution to the operator equation (1) with forcing term

$(A(\mathbf{y}) - A^{(B)}(\mathbf{y}))u^{(B)}(\mathbf{y})$ . Consequently, using the inf – sup conditions twice and the decay property (62), we obtain

$$\|u(\mathbf{y}) - u^{(B)}(\mathbf{y})\|_{\mathcal{X}} \leq \frac{C}{\mu} \left\| A(\mathbf{y}) - A^{(B)}(\mathbf{y}) \right\|_{\mathcal{L}(\mathcal{X}, \mathcal{Y}')} \|u^{(B)}(\mathbf{y})\|_{\mathcal{X}} \leq \frac{\varepsilon C}{\mu} \|f\|_{\mathcal{Y}'},$$

which concludes the proof.  $\square$

Consequently, it is sufficient to draw the  $m_l$  samples per level at random according to the truncated distribution. As a concrete example let us consider the case of linear dependence on the parameters as described in [48] and in Eq. (5). Assuming that  $A_0 : \mathcal{X} \rightarrow \mathcal{Y}'$  is invertible (which was required in Theorem 3) and that  $(b_{0,j})_j \in \ell_1$  (which is weaker than the conditions in the previous section) the fluctuations  $A_j$ ,  $j \geq 1$  are arranged in nonincreasing order, i.e., such that  $b_{0,j} \geq b_{0,k}$  for  $1 \leq j \leq k$ , then the operator (5) satisfies the following dimension truncation error bound

$$\|A(\mathbf{y}) - A^{(B)}(\mathbf{y})\|_{\mathcal{L}(\mathcal{X}, \mathcal{Y}')} = \left\| \sum_{j>B} y_j A_j \right\|_{\mathcal{L}(\mathcal{X}, \mathcal{Y}')} = \|A_0\|_{\mathcal{L}(\mathcal{X}, \mathcal{Y}')} \sum_{j>B} b_{0,j},$$

for any  $\mathbf{y} \in U$ . Moreover (see [48, Thm 2.9], [43, Thm 5.1]), the tail can be estimated by

$$\sum_{j>B} b_{0,j} \leq \min \left\{ \frac{1}{1/p - 1}, 1 \right\} \| (b_{0,j})_j \|_p B^{-(1/p-1)}$$

for some  $p < 1$ . Consequently, choosing  $B \geq h_L^{-(t+t')p/(1-p)}$  yields a global approximation (accounting for the truncation error, the PG approximation error, and the CS error) in  $\mathcal{O}(h_L^{t+t'})$ .

**5.2. Initial set of candidate vectors.** As detailed in the discussion before Theorem 3, the results are, so far, developed for an infinite Chebyshev expansion. To render the problem computationally feasible, we truncate to a finite-dimensional, parametric expansion, where the truncation dimension is at our disposal and therefore can be considered a discretization parameter. Let the sums (34) and (35) be truncated to a finite set  $\Gamma_l \subset \mathcal{F}$ . Some strategies for selecting such a set  $\Gamma_l$  were already described in [48], which was based on the work in [49]. We have the following analog to Theorem 2 (proven in [49]) in the case of expansions in terms of a countable sequence of parameters.

**Theorem 8.** *Let  $\gamma \in (0, 1)$ . Let  $F(\mathbf{y}) = \sum_{\nu \in \mathcal{F}} F_\nu T_\nu(\mathbf{y})$  be a function with  $\|\mathbf{F}\|_{\omega, p} = \|(F_\nu)_\nu\|_{\omega, p} < \infty$  for some  $p < 1$  and some weights  $\omega_\nu \geq \|T_\nu\|_\infty$  for all  $\nu \in \mathcal{F}$ . For a given sparsity  $s_l \geq 1$ , define the initial set of indices as*

$$\Gamma_l := \{\nu \in \mathcal{F} : \omega_\nu^2 \leq s_l/2\}. \quad (64)$$

Furthermore, assume that  $N_l := |\Gamma_l|$  is finite and draw

$$m_l \geq c_0 s_l \max\{\log^3(s_l) \log(N_l), \log(1/\gamma)\} \quad (65)$$

sampling points  $\mathbf{y}^{(i)}$  independently and identically distributed according to the orthogonalization measure  $\eta$ . Let  $\widehat{\mathbf{F}}$  be the solution of

$$\min \|\mathbf{H}\|_{\omega, 1} \quad \text{subject to } \|\mathbf{A}\mathbf{H} - b\|_2 \leq 2^{1-p} \tau \sqrt{m_l} s_l^{1/2-1/p} \|\mathbf{F}\|_{\omega, p},$$

for some  $\tau \geq 1$  and set  $\widehat{F} = \sum_{\nu \in \Gamma_l} \widehat{\mathbf{F}}_\nu T_\nu$ . Then, with probability at least  $1 - \gamma$

$$\begin{aligned} \|F - \widehat{F}\|_\infty &\leq \|\mathbf{F} - \widehat{\mathbf{F}}\|_{\omega, 1} \leq c_\tau s_l^{1-1/p} \|\mathbf{F}\|_{\omega, p}, \\ \|F - \widehat{F}\|_2 &= \|\mathbf{F} - \widehat{\mathbf{F}}\|_2 \leq d_\tau s_l^{1/2-1/p} \|\mathbf{F}\|_{\omega, p}. \end{aligned}$$

A drawback of the recovery based on an optimization problem is that it requires the knowledge (or an approximation) of the norm of the unknown vector  $\mathbf{F}$ . This can be overcome in practical applications by applying the recovery to various estimations (similar to a cross validation in the machine learning literature [55]) or by using greedy methods, e.g. [15, 31].

The cardinality  $N_l$  of the set

$$\Gamma_l = \{\nu \in \mathcal{F} : \omega_\nu^2 \leq s_l/2\} = \{\nu \in \mathcal{F} : \|\nu\|_0 \log(\theta) + \sum_{j \in \text{supp } \nu} 2 \log(v_j) \nu_j \leq \log(s_l/2)\},$$

where the weights  $\omega_\nu$  are chosen as in (44), influences the number  $m_l$  of samples in (65) (and the computational complexity of the weighted  $\ell_1$ -minimization problem). Obviously,  $N_l$  depends on  $s_l$  as well as on the weight sequence  $(v_j)$  used in the definition (44) of  $(\omega_\nu)$ . We recall the following estimates from [48].

**Proposition 3.** *Let  $\omega_\nu = \theta^{|\nu|} \mathbf{v}^\nu$ ,  $\nu \in \mathcal{F}$ , for a sequence  $\mathbf{v} = (v_j)_{j \geq 1}$  specified below and assume  $s_l \geq 1$ .*

- (1) *For  $v_j = \beta$  for  $1 \leq j \leq d$  and  $v_j = \infty$  for  $j > d$  (i.e., we consider constant weights for the first  $d$  dimensions and ignore the remaining ones), we have*

$$N_l = |\Gamma_l| \leq \begin{cases} \left( \left(1 + \frac{1}{\log_2(\beta^2)}\right) ed \right)^{\log_{2\beta^2}(s_l/2)}, & s_l < 2^{d+1}\beta^{2d}, \\ (\log_{\beta^2}(\beta^2 s_l/2))^d, & s_l \geq 2^{d+1}\beta^{2d}. \end{cases}$$

- (2) *For polynomially growing weights  $v_j = cj^\alpha$  with  $c > 1$  and  $\alpha > 0$ , there holds subexponential growth*

$$N_l \leq C_{\alpha,c} s_l^{\gamma_{\alpha,c} \log(s_l)}$$

for some constants  $C_{\alpha,c} > 0$  and  $\gamma_{\alpha,c} > 0$  depending only on  $c$  and  $\alpha$ .

Inserting these bounds into Condition (65) on the number of required samples (assuming that the  $\log(1/\gamma)$ -term does not exceed the other logarithmic terms) shows that the following choices of  $m_l$  are valid:

- For constant weights  $v_j = \beta$  for  $1 \leq j \leq d$  and  $v_j = \infty$  for  $j > d$ , we can chose

$$m_l \asymp \begin{cases} \log(d) s_l \log^4(s_l), & s_l < 2^{d+1}\beta^{2d}, \\ ds_l \log^3(s_l) \log(\log(s_l)), & s_l \geq 2^{d+1}\beta^{2d}. \end{cases} \quad (66)$$

- For polynomially growing weights  $v_j = cj^\alpha$  with  $c > 1$  and  $\alpha > 0$ , we can chose

$$m_l \asymp s_l \log^5(s_l). \quad (67)$$

The case of exponentially growing weights has been analyzed in [48] and yields situations where  $N_l \leq m_l$ . In this situation, compressed sensing techniques should not be used, as least-squares methods are expected to perform better [45].

We note that in the case of constant weights, the first case in (66) is the most relevant. In fact, with the choice of  $s_l$  as in (56), i.e.,  $s_l = \mathcal{C} 2^{(L-l)(t+t')p/(1-p)}$  for some proportionality constant  $\mathcal{C} > 0$ , if  $c := \frac{d+1+2d \log_2(\beta)}{(t+t')p} (1-p) - \frac{\log_2(\mathcal{C})(1-p)}{(t+t')p}$  is large enough (for instance  $c \geq L$ , which is true whenever  $\mathcal{C} \leq s^{d+1}\beta^{2d}/2^{L(t+t')p/(1-p)}$ ) then only the first case of (66) will occur for all  $l = 1, \dots, L$ . In particular, with all the parameters ( $\beta$ ,  $t$ ,  $t'$ , and  $p$ ) fixed, a larger number  $d$  of active variables will lead to a larger  $c$ . It is therefore reasonable to assume that this corresponds to the main regime.

**5.3. Computational Cost.** In the ensuing work bounds, we assume at our disposal multigrid solvers as described, e.g. in [37, 56]. These solvers compute approximate solutions of the Galerkin equations at cost scaling linearly in the number of unknowns of the mesh. This gives rise to the following complexity estimates, where we treat the case of constant and polynomially growing weights.

**Proposition 4.** *Under the assumptions (64), (65) as well as (38), (39) for some  $0 < p < 1$  and smoothness parameters  $t, t'$ , the function  $\mathbf{y} \mapsto \mathcal{G}(u(\mathbf{y}))$  can be approximated in  $L^2(U, \eta)$  to accuracy  $\mathcal{O}(h_L^{t+t'})$  via a MLCSPG discretization with  $L$  levels and with total work  $W_L^T$  scaling as*

$$W_L^T \lesssim \begin{cases} \frac{\log(d) \sigma_p(\tau)^4 L^4 2^{L\sigma_p(\tau)}}{\sigma_p(\tau) - n}, & \sigma_p(\tau) > n, \quad (v_j) \text{ constant} \\ \frac{\log(d) \sigma_p(\tau)^4 L^5 2^{nL}}{\log(d) \sigma_p(\tau)^4 (2^{nL} - 2^{\sigma_p(\tau)L})}, & \sigma_p(\tau) = n, \quad (v_j) \text{ constant} \\ \frac{(n - \sigma_p(\tau))^4}{\sigma_p(\tau)^5 L^5 2^{L\sigma_p(\tau)}}, & \sigma_p(\tau) < n, \quad (v_j) \text{ constant} \\ \frac{\sigma_p(\tau) - n}{\sigma_p(\tau)^5 L^5 2^{L\sigma_p(\tau)}}, & \sigma_p(\tau) > n, \quad (v_j) \text{ polynomial} \\ \frac{\sigma_p(\tau) - n}{\sigma_p(\tau)^5 L^6 2^{nL}}, & \sigma_p(\tau) = n, \quad (v_j) \text{ polynomial} \\ \frac{\sigma_p(\tau)^5 (2^{nL} - 2^{\sigma_p(\tau)L})}{(n - \sigma_p(\tau))^5}, & \sigma_p(\tau) < n, \quad (v_j) \text{ polynomial} \end{cases} \quad (68)$$

where  $\sigma_p(\tau) = \tau p / (1 - p)$  with  $\tau = t + t'$  and where  $n$  denotes the spatial dimension.

*Proof.* Multigrid solvers have a computational complexity scaling linearly with the number  $w_l \asymp 2^{nl}$  of unknowns at level  $l$  which implies that the work at level  $l$  is on the order of  $W_l = m_l \cdot w_l$ ,  $1 \leq l \leq L$ .

Assuming we are given constant weights  $v_j = \beta$ , for  $1 \leq j \leq d$ , and  $s_l < 2^{d+1}\beta^{2d}$ , and that  $d$  is sufficiently large, we can chose  $m_l$  as in the first row of Eq. (66). Thus, omitting constants,

$$\begin{aligned} W_L^T &= \sum_{l=1}^L W_l \lesssim \sum_{l=1}^L \log(d) s_l \log^4(s_l) 2^{nl} \lesssim \sum_{l=1}^L \log(d) 2^{(L-l)\sigma_p(\tau)} \log^4\left(2^{(L-l)\sigma_p(\tau)}\right) 2^{nl} \\ &\lesssim \log(d) \sigma_p(\tau)^4 2^{nL} \sum_{l=1}^L (L-l)^4 2^{(L-l)(\sigma_p(\tau)-n)} = \log(d) \sigma_p(\tau)^4 2^{nL} \sum_{j=1}^{L-1} j^4 2^{j(\sigma_p(\tau)-n)}. \end{aligned} \quad (69)$$

We can bound  $S := \sum_{j=1}^{L-1} j^4 2^{j(\sigma_p(\tau)-n)} \leq \int_0^L 2^{x(\sigma_p(\tau)-n)} x^4 dx$ . If  $\sigma_p(\tau) = n$ , it follows that  $S \leq L^5/5$ . Otherwise, with  $K = (\sigma_p(\tau) - n) \ln(2)$ , an integration by part yields

$$S \leq \frac{L^4 e^{LK}}{K} - \frac{4}{K} \int_0^L x^3 e^{xK} dx. \quad (70)$$

If  $K > 0$ , i.e.  $\sigma_p(\tau) > n$ , the remaining integral is positive and thus  $S \leq \frac{L^4 e^{LK}}{K} = \frac{L^4 2^{L(\sigma_p(\tau)-n)}}{\ln(2)(\sigma_p(\tau)-n)}$ . If  $K < 0$ , repeated integration by parts leads to

$$S \leq \frac{L^4 e^{LK}}{K} - \frac{4L^3 e^{LK}}{K^2} + \frac{12L^2 e^{LK}}{K^3} - \frac{24L e^{LK}}{K^4} + \frac{24}{K^4} \int_0^L e^{xK} dx. \quad (71)$$

Noticing that  $\frac{L^4 e^{LK}}{K} - \frac{4L^3 e^{LK}}{K^2} + \frac{12L^2 e^{LK}}{K^3} - \frac{24L e^{LK}}{K^4} < 0$ , it follows that

$$S \leq \frac{24}{K^4} \int_0^L e^{xK} dx = 24 \frac{e^{LK} - 1}{K^5} = \frac{24(1 - 2^{(\sigma_p(\tau)-n)L})}{(n - \sigma_p(\tau))^5 \ln(2)^5}. \quad (72)$$

The result for polynomially growing weight sequences ( $v_j$ ) is shown in a similar fashion (with appropriate changes in exponents).  $\square$

**Remark 2.** Recalling that the workload for the computation of one solution at the finest discretization level  $L$  is  $w_L \asymp 2^{nL}$ , the previous result means that for  $\sigma_p(t+t') < n$ , the total work is bounded only by a multiple of the cost of one PDE solve at the finest level, where the multiplicative constant involves a factor of  $\log(d)$  in the case of constant weights and in addition only depends on  $n, p, t, t'$ .

Combining Theorem 3 together with Proposition 4 about the computation costs and Proposition 2 regarding the truncation of the operator, we are finally able to state our main theorem. To this end we first summarize the assumptions on the parametric operator  $A(\mathbf{y}) = A_0 + \sum_{j \geq 1} y_j A_j$ .

- The nominal operator  $A_0$  is inf-sup stable, i.e.,

$$\inf_{0 \neq w^h \in \mathcal{X}^h} \sup_{0 \neq v^h \in \mathcal{Y}^h} \frac{\langle A_0 w^h, v^h \rangle}{\|w^h\|_{\mathcal{X}} \|v^h\|_{\mathcal{Y}}} \geq \mu_0 > 0, \quad \inf_{0 \neq v^h \in \mathcal{Y}^h} \sup_{0 \neq w^h \in \mathcal{X}^h} \frac{\langle A_0 w^h, v^h \rangle}{\|w^h\|_{\mathcal{X}} \|v^h\|_{\mathcal{Y}}} \geq \mu_0 > 0.$$

- For some  $0 < p < 1$  and some weight sequence  $\mathbf{v} = (v_j)_{j \in \mathbb{N}}$  with  $v_j \geq 1$ , the sequence  $\mathbf{b}_0$  with components  $b_{0,j} = \|A_0^{-1} A_j\|_{\mathcal{L}(\mathcal{X})}$ ,  $j \geq 1$ , satisfies

$$\kappa_{\mathbf{v},p} := \sum_{j \geq 1} b_{0,j} v_j^{(2-p)/p} < 1 \quad \text{and} \quad \sum_{j \geq 1} b_{0,j}^p v_j^{2-p} < \infty.$$

- For some  $t \in (0, \bar{t})$ , the operators  $A_j$ ,  $j \geq 0$ , are defined as operators from  $\mathcal{X}_t$  into  $\mathcal{Y}_t$  the sequence  $\mathbf{b}_t$  with components  $b_{t,j} = \|A_0^{-1} A_j\|_{\mathcal{L}(\mathcal{X}_t)}$  satisfies

$$\kappa_t := \sum_{j \geq 1} b_{t,j} \leq 1, \quad \mathbf{b}_t \in \ell^{pt}.$$

**Theorem 9.** *Let  $L \in \mathbb{N}$  be a number of discretization levels and  $\gamma_\ell \in (0, 1)$ ,  $\ell = 1, \dots, L$ . Let  $A(\mathbf{y})$  be an affine-parametric operator and let  $\mathbf{v} = (v_j)_{j \geq \mathbb{N}}$  be a sequence of weights with  $v_j \geq 1$ . Assume that  $A_0 \in \mathcal{L}(\mathcal{X}_t, \mathcal{Y}'_t)$  is boundedly invertible and that the sequence  $\mathbf{b}_t = (b_{t,j})_{j \geq 1}$  are such that the summability conditions (41) and (39) hold true for some  $0 < p < 1$ . Then, for any discretization level  $1 \leq l \leq L$ , the sequence of Chebyshev coefficients of  $\Delta u^l$  with respect to the parameter vector (34) is (weighted) compressible, i.e., for a sequence of weights  $\omega = (\omega_\nu)_{\nu \in \mathcal{F}}$  with  $\omega_\nu = \theta^{\|\nu\|_0} \mathbf{v}^\nu$  there holds  $\sum_{\nu \in \mathcal{F}} \omega^{2-p} \|\Delta u_\nu^l\|_{\mathcal{X}}^p < \infty$ .*

*Moreover, if we are interested in a functional of the solution  $F(\mathbf{y}) = \mathcal{G}(u(\mathbf{y}))$  and if the operators  $A(\mathbf{y}) \in \mathcal{L}(\mathcal{X}_t, \mathcal{Y}'_t)$  are boundedly invertible in the smoothness scales  $(\mathcal{X}_t, \mathcal{Y}_t)$  in (14), (15) and if  $\mathcal{G} \in \mathcal{X}'_t$ , then the function  $F(\mathbf{y}) = \sum_{\nu \in \mathcal{F}} F_\nu T_\nu(\mathbf{y})$  can be approximated by  $F_{MLCS}^L(\mathbf{y}) := \sum_{l=1}^L \widehat{\Delta F}^l(\mathbf{y})$  where  $\widehat{\Delta F}^l(\mathbf{y})$  is a single-level CSPG approximation from*

$$m_l \asymp s_l \max\{\log^3(s_l) \log(N_l), \log(1/\gamma_l)\}$$

*sampling points with  $s_l \asymp 2^{(L-l)(t+t')p/(1-p)}$ , where  $N_l = |\Gamma_l|$  for  $\Gamma_l = \{\nu \in \mathcal{F} : \omega_\nu^2 \leq s_l/2\}$ .*

*Then, with probability at least  $1 - \sum_{l=1}^L \gamma_l$ , this approximation fulfills the bounds*

$$\|F - F_{MLCS}^L\|_\infty \leq Ch_L^{t+t'} \|f\|_{\mathcal{Y}'_t} \|\mathcal{G}\|_{\mathcal{X}'_t} (L+1), \quad (73)$$

$$\|F - F_{MLCS}^L\|_2 \leq C' h_L^{t+t'} \|f\|_{\mathcal{Y}'_t} \|\mathcal{G}\|_{\mathcal{X}'_t}, \quad (74)$$

*and can be computed in a total work that scales as*

$$W_L^T \lesssim \begin{cases} 2^{nL}, & \sigma_p(\tau) < n, \\ L^{\xi+1} 2^{nL}, & \sigma_p(\tau) = n, \\ L^\xi 2^{L\sigma_p(\tau)}, & \sigma_p(\tau) > n, \end{cases} \quad (75)$$

*where  $\xi = 4$  or  $5$  for constant or polynomially growing weights  $\mathbf{v}$ , respectively.*

*Proof.* This theorem follows from applying Theorem 8 at each level  $l \in \{1, \dots, L\}$  with probability of failure  $\gamma_l$  and taking the union bound. The bound (73) follows from Theorem 3 and using the calculations in Prop.4.  $\square$

Note that we can make the failure probability more explicit. Choosing  $s_l$  as in (56), i.e.,  $s_l = C2^{\sigma_p(L-l)}$  with  $\sigma_p = \frac{(t+t')p}{1-p}$  and equating both terms in the max defining the number of samples so that

$$m_l \asymp s_l \log(s_l)^3 \log(N_l) \asymp \sigma_p^3 2^{\sigma_p(L-l)} (L-l) \log(N_l), \quad l = 1, \dots, L$$

gives

$$\gamma_l = N_l^{-\log^3(s_l)} = N_l^{-(c_1 \sigma_p(L-l) + c_2)^3}, \quad l = 1, \dots, L.$$

This results in a probability of failure at most

$$\begin{aligned} \sum_{l=1}^L \gamma_l &= \sum_{l=1}^L N_l^{-\log^3(s_l)} = \sum_{l=1}^L N_l^{-(c_1 \sigma_p(L-l) + c_2)^3} \leq \sum_{l=1}^L N_L^{-(c_1 \sigma_p(L-l) + c_2)^3} \lesssim N_L^{-c_2^3} \sum_{l=1}^L N_L^{-(c_1 \sigma_p^3(L-l))^3} \\ &\lesssim N_L^{-c_3}, \end{aligned}$$

where it is used that  $N_1 \geq N_2 \geq \dots \geq N_L$  by definition of  $\Gamma_l$  and  $s_l$ .

## 6. NUMERICAL RESULTS

In this section, we illustrate our theoretical findings with some numerical examples. All these examples are implemented in Python 2.7. For the PDE solves we use the tools developed via the FEniCS project [2, 44].<sup>3</sup> The sparse recovery problem is either solved by greedy approaches using functions developed in house or using CVXPY [23] as convex solver for the weighted  $\ell_1$  problem. We use degree one Lagrange polynomials as finite elements.

<sup>3</sup>Note that all the code for reproducible research and further use is available from one of the authors' github page: <https://github.com/jlbouchot/CSPDES>

**6.1. Convergence.** We start by looking at the convergence of the approach with respect to the meshwidth. To this end, we fix the number of levels used for the MLCSPG approximation to  $L = 3$  and let the coarser meshwidth vary from  $h_0 = 1/5$  to  $h_0 = 1/70$ . We want to illustrate the results suggested by Theorem 9. We consider the diffusion problem (3) and the QoI  $F(\mathbf{y}) = \int_{x \in D} u(x, \mathbf{y}) dx$ . The diffusion coefficient is represented via a cosine expansion, i.e., we set

$$a(x, \mathbf{y}) = \bar{a}(x) + \sum_{j=1}^d y_j \frac{\cos(\pi j \|x\|)}{j^\mu}, \quad x \in \mathbb{R}^n, \mathbf{y} \in [-1, 1]^d. \quad (76)$$

For the results presented here, we choose  $n = 1, 2, 3$ ,  $\mu = 2$ , and  $d = 10, 15, 20$ . We set uniform weights  $v_j = 1.08$  and  $\omega_\nu = \theta^{\|\nu\|_0} \mathbf{v}^\nu$  as suggested in Theorem 9. The mean field and forcing term are kept constant  $\bar{a} \equiv 4.3$  and  $f \equiv 10$ , respectively.

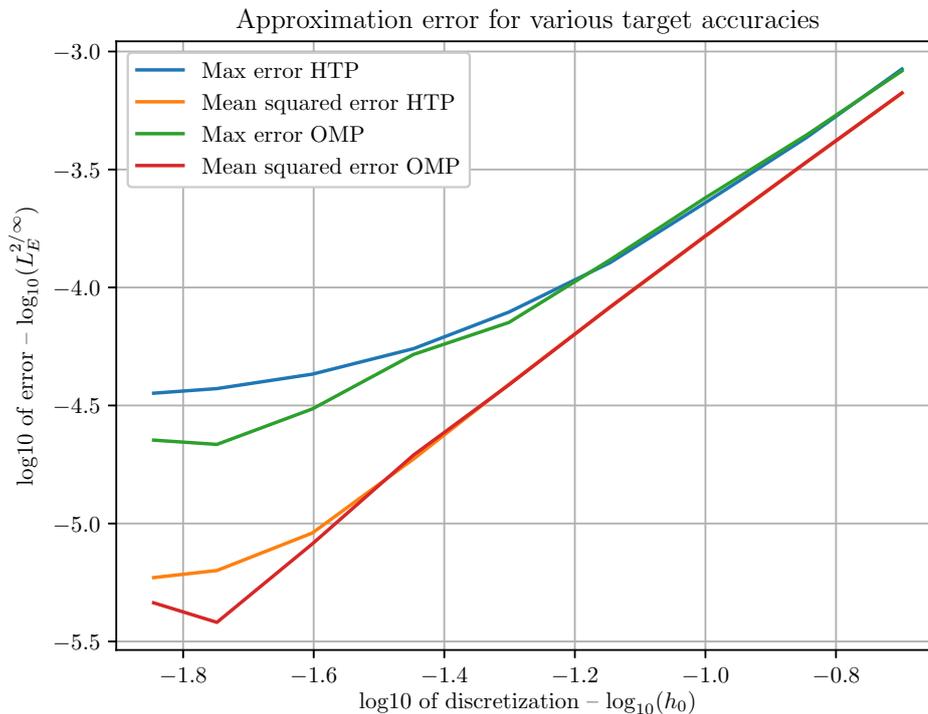


FIGURE 1. Convergence of the MLCSPG method for the diffusion problem with cosine expansion of the diffusion coefficient.  $d = 20$ . The graph shows  $\log(\text{error})$  as a function of  $\log(h_0)$  where the empirical error is measured in the  $L_E^2$  and  $L_E^\infty$  norm and for sparse recovery done with either HTP or OMP. See text for more details.

Figure 1 illustrates the convergence of the method with respect to the coarser meshwidth  $h_0$  in  $n = 2$  spatial dimensions and  $d = 20$  parameters. The empirical errors are calculated as

$$\begin{aligned} L_E^1 &:= \frac{1}{N_{\text{test}}} \sum_{1 \leq j \leq N_{\text{test}}} |F(\mathbf{y}^{(j)}) - F^{\text{CSPG}}(\mathbf{y}^{(j)})|, \\ L_E^2 &:= \sqrt{\frac{1}{N_{\text{test}}} \sum_{1 \leq j \leq N_{\text{test}}} |F(\mathbf{y}^{(j)}) - F^{\text{CSPG}}(\mathbf{y}^{(j)})|^2}, \\ L_E^\infty &:= \max_{1 \leq j \leq N_{\text{test}}} |F(\mathbf{y}^{(j)}) - F^{\text{CSPG}}(\mathbf{y}^{(j)})|, \end{aligned}$$

for  $N_{\text{test}} = 1000$  independent draws of random parameter vectors  $\mathbf{y}$  and where the ground truth used for comparison is a numerical approximation computed on a grid that is at least 4 times finer. The sparse recovery methods used for this figure are iterative (HTP [33], picked for its proven fast convergence [10]) and greedy (OMP) approaches. The finite element method is used with a degree one polynomial and with an iterative Krylov solver for the inversion of the system involving the stiffness matrix.

The (level dependent) number of samples and sparsities have been chosen as

$$m_l = 2 \cdot s_l \log(N_l), \quad (77)$$

$$s_l = 8 \cdot 2^{L-l}, \quad (78)$$

where  $N_l = |\Gamma_l|$  is the set of Chebyshev polynomials truncated according to Eq. (64). The choice of  $m_l$  differs slightly from the theoretically justified choice in Eq. (57). The selection (77) refers to the usual rule of thumb in compressed sensing which is justified by non-uniform recovery results with random matrices, see [34, Ch.9.2] for details. While the choice (77) of numbers of CS sample  $m_l$  is below what is sufficient according to our theoretical results, we shall see in the numerical examples ahead that even this optimistic selection of sample number is more than sufficient for our problems. The choice (78) of  $s_l$  corresponds to Eq. (56) where the proportionality constant is chosen as 8 and the regularity assumption of the solution is taken as  $\sigma_p(t+t') = (t+t')p/(1-p) = 1$  to simplify the exposition. This constant could be estimated numerically from Figure 1.

**6.2. Computing times.** We now investigate the actual computational complexity required for our approach. We consider the same framework as in the previous section but consider  $d = 15$  parameters in  $n = 2$  and 3 spatial dimensions. All other parameters are kept the same.

Figure 2 shows the ratio of the time required for the sparse recovery problems against the time required for computing the PDE solutions. The times (for sparse recovery and PDE solves) are reported by adding all contributions at every level and then the ratio sparse recovery to PDE solves is graphed. We displayed the results when using weighted versions of OMP and HTP which should be preferred in high-dimensions. As the complexity increases (red and green curves for the case of  $n = 3$  spatial dimensions) the time required for the sample evaluations becomes more and more important compared to the time needed for the sparse recovery. Note that the orange curve displaying the computing time for the weighted OMP in  $n = 2$  dimensions, while being cut at the top, never reaches more than 0.35 in our experiments. Table 1 shows a precise description of the time required for the recovery and sample evaluations at each level when considering  $d = 15$  parameters and two different original meshsizes  $h_0 = 1/5$  and  $h_0 = 1/40$ . As it can be seen, the efficiency of the presently proposed MLCSPG approach increases for more expensive forward solves, i.e. with increasing complexity of the simulated system. Moreover, it is important to point out that the sparse recovery component is completely independent of the size of the spatial dimension and its discretization as illustrated in Table 1.

**6.3. Single-level versus Multi-level CSPG.** To ensure the necessity of the multi-level approach developed in this paper, we compare with the original single-level method introduced by two of the named authors [48]. We report on the computational time required to reach a given accuracy, parametrized by fixing  $h_L$ . We compare the single-level approach ( $L = 1$ ) with multi-level schemes  $L = 2, 3$ . By varying the constants of proportionality related to the sparsity per level (see Eq. (78)) we can control the computational time. Fig 3 illustrates this behavior when dealing with  $d = 15$  parameters by plotting the logarithm of the

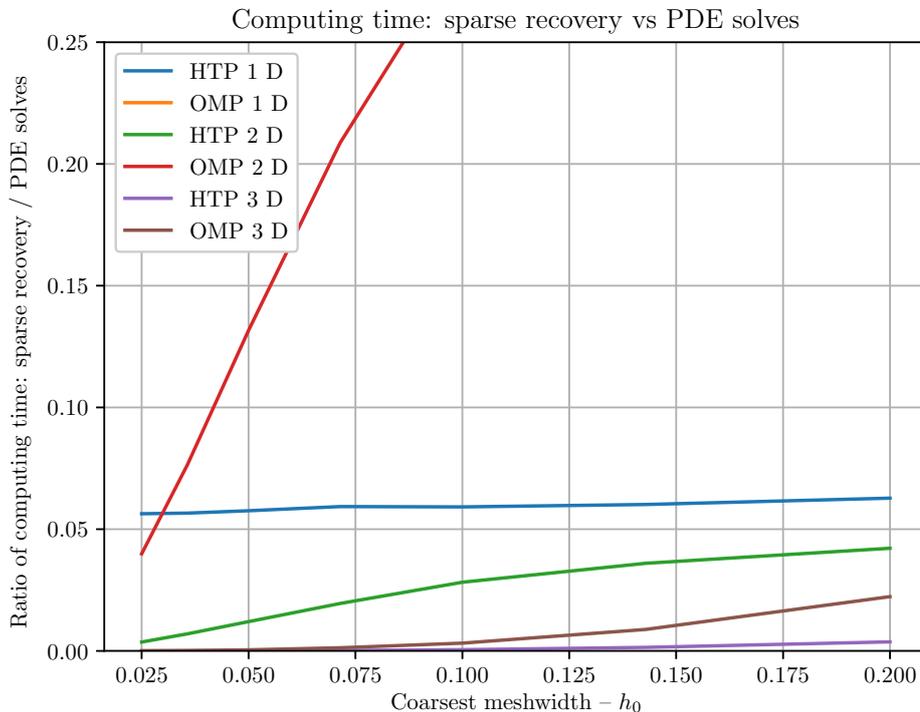


FIGURE 2. Ratio of time for the computations for the sparse recovery against the time for the PDE solves. As the model gets more and more complex, the computing time for the sparse recovery is more and more negligible.

TABLE 1. Comparison of the different computing times for various settings (in seconds)

	$l = 0$				$l = 1$				$l = 2$			
	HTP		PDE		HTP		PDE		HTP		PDE	
$h_0 =$	1/5	1/40	1/5	1/40	1/5	1/40	1/5	1/40	1/5	1/40	1/5	1/40
$n = 1$	2.8876	2.8879	24.324	24.787	0.1730	0.1156	19.651	20.186	0.0111	0.0108	7.6102	8.0154
$n = 2$	2.9699	2.8923	32.247	158.59	0.2051	0.2048	27.987	286.88	0.0129	0.0926	15.339	428.39
$n = 3$	2.9079	2.8167	84.965	24221	0.2034	0.1946	207.31	88919	0.0119	0.0961	569.01	286804

computing time (including both the sample evaluations and the sparse recovery parts) with respect to the (log of the) accuracy.

Once again we consider the parametric diffusion problem from Eq. (3) in  $n = 2$  spatial dimensions and with the cosine expansion of the diffusion coefficient as in Eq. (76). The cosine coefficients have decay parameter  $\mu = 2$  and the mean-field is set to be a constant coefficient  $\bar{a}(x) = 4.3$ , for all  $x \in [0, 1]^2$ .

The proportionality constant in (78) (where it is 8) varies from 5 to 14 for the single-level case, from 3 to 8 for the  $L = 2$  case, and from 3 to 6 for the  $L = 3$  case. First, as suggested by the SLCSPP theory (see [48, 9]), the single level approach is indeed limited by the quality of the approximation grid and eventually reaches a maximum accuracy. At this point, adding more samples does not yield better approximation capabilities (see the blue and orange curves). On the other hand, using the multi-level approach, we can reach better accuracy while reducing the computations, illustrated by the fact that the green, red, purple and brown curves are all below and to the left of the SLCSPP curves (blue and orange curves in Fig. 3).

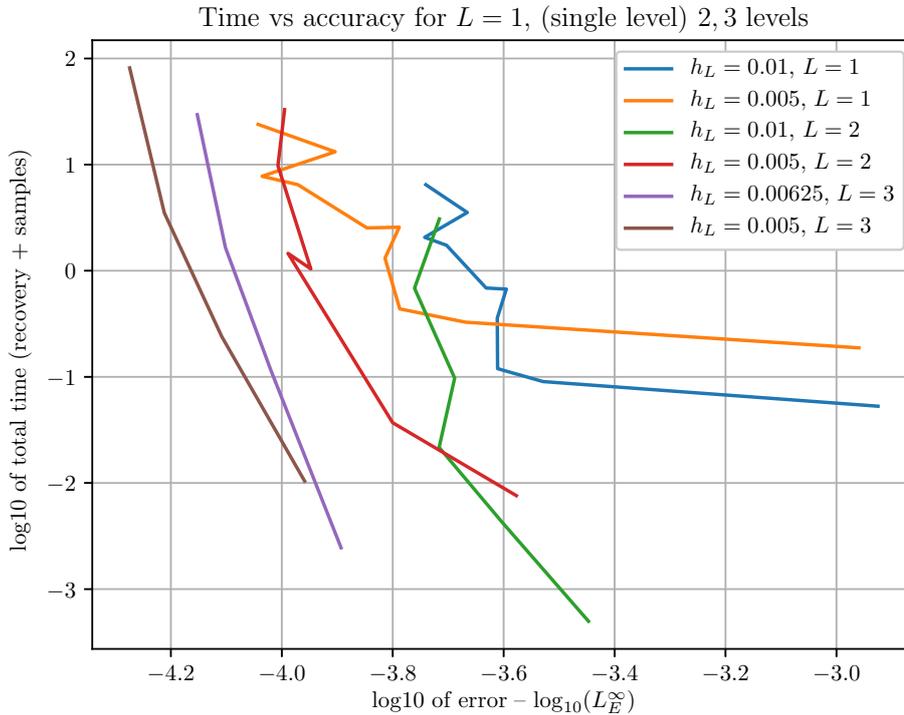


FIGURE 3. Time for evaluating the samples and sparse recovery against accuracy for  $L = 1, 2, 3$  levels of approximations. The case  $L = 1$  corresponds to the SLCSPPG approach.

Comparing the curves for the highest resolution plotted ( $h_L = 0.005$ , orange for the single level computations, red for two levels, and brown for the three level case), the advantage in terms of computations and accuracy of the presented methods becomes noticeable. Interestingly, the use of a multi-level procedure seems to break (to some extent) the mesh-size barrier imposed by the single-level – a remark worth investigating further.

**6.4. Variations with the dimension.** To ensure the numerical scalability of the approach as the parametric dimension increases, we ran tests on a piecewise constant diffusion problem in two spatial dimensions. In this set of experiments we partition the spatial domain  $[0, 1]^2$  into  $d = 3^2, 4^2, 5^2$ , and  $6^2$  equal patches  $D_j$ ,  $1 \leq j \leq d$ . The diffusion coefficient from Eq. (3) is given by

$$a(x, \mathbf{y}) = \bar{a}(x) + v \sum_{j=1}^d y_j \chi_{D_j}(x), \quad (79)$$

where  $\chi_{D_j}$  corresponds to the characteristic function of the patch  $D_j$ . We report our experiments in Table 2, where the local variations have an amplitude up to  $v = 2$  and the mean field is constant  $\bar{a} \equiv 5$ . This table shows the time required for computing the PDE solutions as well as the sparse recovery procedures (done via the weighted version of HTP here) for the different numbers of parameters. We also mention the number of samples and the size of the truncated active set  $\Gamma_l$  at every level.

As claimed in this article, we are capable of breaking the curse of dimensionality. Indeed, computing the ratio of the logarithms of the computing times to the dimension shows that complexity only scales polynomially in the number of parameters (as claimed in Theorem 9). Moreover, having set a target accuracy of  $h_L = 2.5 \cdot 10^{-3}$  we verified the accuracy of our recovered solutions against 1000 random independent tests.

TABLE 2. Statistics on the computing time, number of samples and size of active sets, and accuracy for a piecewise constant diffusion problem in two spatial dimensions with various parametric dimensions. These values were obtained using FEniCS as direct forward solver, and with a Python implementation.

Size	$3 \times 3$	$4 \times 4$	$5 \times 5$	$6 \times 6$	
$d$	9	16	25	36	
$L_E^2$	0.000271	0.000478	0.000845	0.001005	
$L_E^\infty$	0.001176	0.001687	0.004229	0.004081	
PDE solves (s)	2991.9141	4304.8436	6049.8758	23437.15047	
Recovery (s)	0.3874	1.7044	5.9584	29.7932	
$\log(\text{time } d_i)/\log(d_i)$	3.6427	3.0180	2.7055	2.8082	
$m$	$l = 0$	1082	1298	1467	1605
	$l = 1$	438	518	582	637
	$l = 2$	165	198	224	246
$N$	$l = 0$	4687	25225	94351	278755
	$l = 1$	931	3241	8851	20731
	$l = 2$	172	473	1076	2143

The ground truth was here numerically approximated on a grid three times finer than the one used for the MLCSPG method.

**6.5. Comparison with  $L^2$  and Monte-Carlo.** We consider the bounded interval  $D = (0, 1)$  with equispaced partition  $\bar{D} = \bigcup_{i=1}^d \bar{D}_i$  into subintervals  $D_i = (x_{i-1}, x_i)$  where  $x_i = i/d$ , for some  $d \in \mathbb{N}$ . We let  $a(x, \mathbf{y}) = \bar{a} + \sum_{j=1}^d y_j c_j \chi_{D_j}(x)$  with  $\bar{a}$  being a constant independent of  $x$ ,  $\{c_j\}_{j=1}^d$  a predefined (fixed) sequence such that the (weighted) uniform ellipticity assumption (6) holds, and  $\chi_{D_j}$  the indicator function of the set  $D_j$ . We select the parameters to be  $d = 6$  and pick uniform (small) local variations as  $c_j = 1/6$ , for  $1 \leq j \leq d$ . The uniform weights  $v_j$  are selected as  $v_j = 1.07$  for all  $j$ . We also set the forcing term  $f \equiv 1$  to be constant. Then, for any  $\mathbf{y} \in U$  the solution to the diffusion equation is continuous and piecewise quadratic. The (level dependent) number of samples and sparsities are chosen as

$$\begin{aligned} m_l &= 2 \cdot s_l \log(N_l), \\ s_l &= 20 \cdot 2^{L-l}. \end{aligned}$$

The initial mesh size is set to  $h_0 = 5 \cdot 10^{-4}$ . Further numerical tests – not included in this paper, but available online – have shown that this parameter has, in this case, little to no influence over the results.

We compare the convergence of our algorithms with other methods: Monte-Carlo sampling and least squares ( $\ell_2$  recovery) [45]. The estimation of the Chebyshev coefficients are displayed in Fig. 4, where the magnitudes of the Chebyshev coefficients of the (functional of the) parametric solution are displayed on a logarithmic (base 10) scale. The  $x$ -axes corresponds to an enumeration of the multi-index of the Chebyshev coefficient, whereby the larger ones (in magnitude, according to the  $\ell_2$  recovery) are first. The least squares solution is obtained as follows. We first build the active set of candidates for the truncated polynomial space as predicted by Theorem 8, i.e.  $\Gamma = \bigcup_{l=1}^3 \Gamma_l$ . This set has total dimension  $N = 12171$ . Then  $m = 24342$  sampling points  $\mathbf{y}^{(i)}$  are chosen at random, and the values  $b_i = F(\mathbf{y}^{(i)})$  are computed and stacked into a vector  $\mathbf{b} = (b_i)_i$ . Finally, the coefficients  $(F_\nu^{\ell_2})_{\nu \in \Gamma}$  are computed as the minimizer of the least squares problem

$$\min_{\mathbf{F}} \|\mathbf{b} - \mathbf{A}\mathbf{F}\|_2,$$

where  $\mathbf{A}_{i,\nu} = T_\nu(\mathbf{y}^{(i)})$ , with  $1 \leq i \leq m$  and  $\nu \in \Gamma$ . To display our results on Fig. 4, we have an (implicit) enumeration  $\pi : \{1, \dots, 12171\} \rightarrow \Gamma$  such that  $|F_{\pi(1)}^{\ell_2}| \geq |F_{\pi(2)}^{\ell_2}| \geq \dots \geq |F_{\pi(12171)}^{\ell_2}|$ . For this experiment, we compute the solutions to the weighted  $\ell_1$ -minimization problems using the SCP optimization procedure from the CVXPY package [23] with accuracy for the numerical optimization set to  $10^{-6}$ . The downward triangles

are the results using our suggested MLCSPG method with the multiplicative constant 8 in Eq. (78) replaced by 5 (red curve) and by 15 (blue curve). The selection of the constant equal to 15 corresponds to  $m_1 = 2258$  solves at the coarsest level,  $m_2 = 968$  at the second, and to  $m_3 = 394$  solves at the finest discretization level  $L = 3$ .  $m_1 = 576$ ,  $m_2 = 228$ , and  $m_3 = 73$  samples, for the red curve. The crosses correspond to the MC simulations, where we have used  $m = 2.5 \cdot 10^7$  (red curve) and  $m = 2.5 \cdot 10^9$  (blue curve) samples for the estimation of the Chebyshev coefficients. Noting that the  $y$  values of the graphs correspond to the  $\log_{10}$  of the magnitude of the coefficients, we see that the accuracy of the MC estimations is limited by the mean square convergence rate  $m^{-1/2}$ . The purple circles correspond to the  $\ell_2$  estimation described above (this corresponds to an oversampling ratio of 2, which is far below theoretical results). In this example, our approach (as illustrated by the downward triangle curves in Fig. 4) produces reliable approximations of gpc coefficients which are large in magnitude with a number of samples orders of magnitudes smaller than both the  $\ell_2$  and the MC approaches. The limitation of the MC method to a square root convergence rate requires a prohibitive number of samples for more complicated PDEs. It is important to note also that the accuracy of the recovered coefficients via our MLCSPG method are constrained by the accuracy of the numerical solver for the weighted  $\ell_1$  minimization. Finally, the yellow curve corresponds to the (negative, for illustrative purposes) total degree of the multi-index of the associated Chebyshev coefficient while the black curve corresponds to the (negative of the) maximum degree in the tensor product (22). It is interesting to notice that the magnitude seems to be smaller as the degree of the multi-index increases.

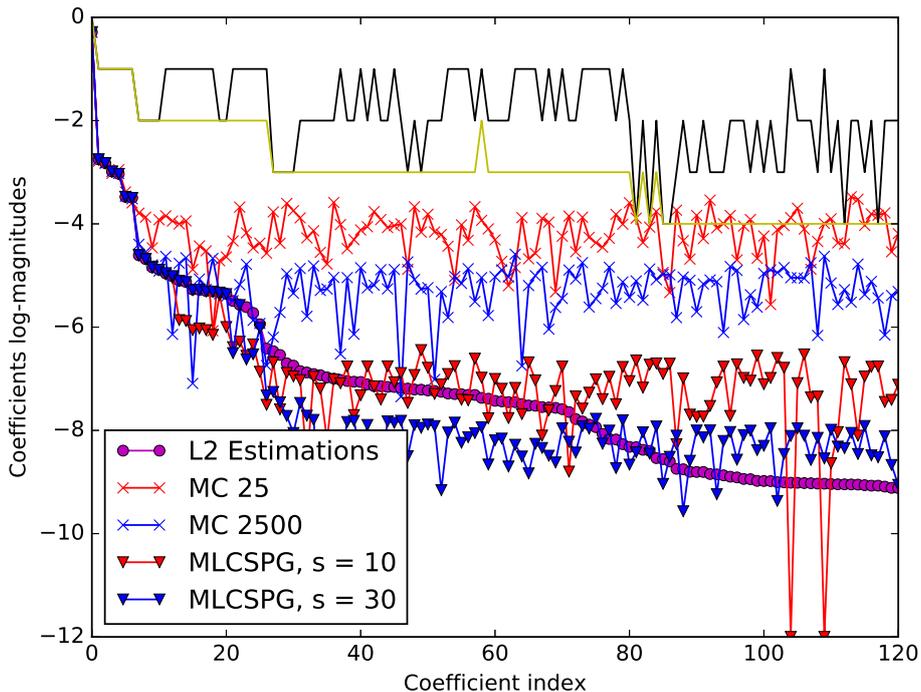


FIGURE 4. Estimations of the 120 largest coefficients (in magnitude) of the gpc of the piecewise constant diffusion problem (see text for details on the parameters) reordered by decreasing magnitude, according to their estimations via a least squares method. The **least squares estimation** computed 12171 coefficients from 24342 random samples; the **downward blue triangles** correspond to a constant 15 in Eq. (78), while the **downward red triangles** correspond to the constant 5. The **MC 25 curve** corresponds to Monte Carlo estimations with  $2.5 \cdot 10^7$  samples while the **MC 2500 curve** is based on  $2.5 \cdot 10^9$  samples.

## 7. CONCLUSIONS

For a class of abstract, affine-parametric, linear operator equations depending on sequences  $\mathbf{y}$  of parameters, we have introduced a multi-level generalization of the CS approach from [48] to efficiently scan the high-dimensional parameter space. For the approximate solution of (instances of) the parametric operator equations, we stipulated available inf-sup stable, Petrov-Galerkin (“PG” for short) discretizations of the “nominal” operator  $A_0 = A(\mathbf{0})$ ; in particular, (46) holds. The small perturbation hypothesis (41) at  $t = 0$  implies uniform (w.r.t.  $\mathbf{y} \in U$ ) inf-sup stability (8) of the PG discretization (Thm. 5). Admissible PG discretizations comprise, in particular, all classical primal or mixed Finite Element Methods (FEM for short), as well as spectral and collocation methods for elliptic and certain linear, parabolic evolution equations. Throughout, we used multi-level Finite Element Galerkin discretizations in  $D \subset \mathbb{R}^n$  with isotropic mesh refinements, responsible for the  $\mathcal{O}(2^{nl})$  scaling in the proof of Proposition 4. Anisotropic, “sparse-grid” discretizations of the parametric problems in  $D$  would result, with analogous analysis, in so-called “multi-index” compressed sensing PG methods, analogous to multi-index MC in [38], with  $\mathcal{O}(l^{n-1}2^l)$  in place of  $\mathcal{O}(2^{nl})$ . We analyzed error vs. work of the multi-level extension of the combined, CS-PG algorithm and showed that it affords improved, as compared to the single-level variant from [48, 9], error vs. work bounds with convergence rates that are independent of the dimension of the space parameters which are active in the approximation, while being “nonintrusive”, i.e. accessing an available solver at each discretization level. This is analogous to what is known from multi-level Monte-Carlo (“MLMC” for short) sampling methods, as surveyed e.g. in [35]. Contrary to MLMC methods whose convergence rate is limited by the (mean-square) rate 1/2 afforded by MC methods, and the recently proposed sparse-grid methods in [16] which rely on a particular (“downward closed”) structure of the sets of active polynomials, however, the presently proposed approach yields dimension-independent convergence rates (potentially far beyond 1/2) in the sup-norm with respect to the parameters, exploiting any sparsity in the gpc coefficient sequence of the parametric solutions, *without* strong, a-priori structural assumptions on the active polynomial degrees. At the same time, the MLCSPG approach is nonintrusive and intrinsically parallel as MLMC methods. If a-priori information on the structure of sets of active indices (such as “downward closedness”) is available, corresponding accelerations of the SLCS approach have recently been investigated in [17]. This is afforded by adopting Chebyshev gpc expansions which are orthonormal with respect to a probability measure which underlies the CS method, whereas sparse-grid methods as in [16] afford greater flexibility as regards the choice of gpc system.

We remark that although here only affine-parametric operator equations were considered, the key results of the present paper require merely *sparsity of Chebyshev gpc expansions* (as expressed, e.g., in summability of sequences of  $\mathcal{X}_t$ -norms of gpc expansion coefficients in the conditions (38) - (40), rather than the weaker summability of  $\mathcal{X}$ -norms in the SLCS approach considered in [48]) of the parametric solutions, and some (possibly crude) bounds of these coefficients which enter the weight sequence  $\omega$ , and a family of uniformly inf-sup stable PG discretization methods. Such results are available for rather general, holomorphic-parametric, nonlinear operator equations in [18]. In case that the  $\psi_j$  in (2), (3) have supports which are localized to subdomains of  $D$  with controlled overlap, higher summability for the Chebyshev gpc expansion coefficients holds; we refer to [3] for details. The presently proposed MLCSPG algorithms are able to exploit better summability of Chebyshev gpc expansion coefficients without any modification in the algorithm.

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CHAIR FOR MATHEMATICS C (ANALYSIS), RWTH AACHEN UNIVERSITY, GERMANY  
*E-mail address:* `bouchot@mathc.rwth-aachen.de`

CHAIR FOR MATHEMATICS C (ANALYSIS), RWTH AACHEN UNIVERSITY, GERMANY  
*E-mail address:* `rauhut@mathc.rwth-aachen.de`

SEMINAR FOR APPLIED MATHEMATICS, ETH ZÜRICH, SWITZERLAND  
*E-mail address:* `christoph.schwab@sam.math.ethz.ch`