Numerical Solution of Underdetermined Systems from Partial Linear Circulant Measurements

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***Abstract*—We consider the traditional compressed sensing problem of recovering a sparse solution from undersampled data. We are in particular interested in the case where the measurements arise from a partial circulant matrix. This is motivated by practical physical setups that are usually implemented through convolutions.**

We derive a new optimization problem that stems from the traditional ℓ_1 minimization under constraints, with the added information that the matrix is taken by selecting rows from a circulant matrix. With this added knowledge it is possible to simulate the full matrix and full measurement vector on which the optimization acts. Moreover, as circulant matrices are well-studied it is known that using Fourier transform allows for fast computations. This paper describes the motivations, formulations, and preliminary results of this novel algorithm, which shows promising results.

I. Introduction

The last decade has seen the emergence of compressed sensing as a technique to recover signals from the knowledge of very few linear samples. Theoretical works and numerical algorithms have been developed in parallel in order to provide solutions to underdetermined linear systems. On the one hand, many research has been done towards the analysis of recovery algorithms based on a so called *Restricted Isometry Property* (RIP) [1][2, Chap.6]. On the other, a lot of effort has been put towards finding matrices fulfilling such RIPs [3], [4], [5]. We concentrate here on the numerical solutions of $Ax = y$ where A is a partial random circulant matrix.

A. Motivation

The compressed sensing community has concentrated on solving very general problems. We try here to restrict ourselves to more applicable solutions. Usual physical sensing devices such as those used, e.g. in MRI machines, implement a convolution, which are themselves modeled in terms of linear measurements by a circulant matrix, or partial Fourier matrices. Though such sensing matrices are important, algorithms are designed in a very general way, independent from the type of measurement process. It is however of prime importance for the applicability of compressed sensing to look at algorithms that are practical and efficient in applications.

The question can now be reformulated into *Given a matrix $A \in \mathbb{R}^{m\times N}$ obtained by (randomly) selecting m rows from a circulant matrix $C \in \mathbb{R}^{N\times N}$ and the knowledge of $y = Ax$ for $y \in \mathbb{R}^m$ and a certain unknown $x \in \mathbb{R}^N$, can we recover, or at least find a suitable approximation of, x?*

B. Contribution and organization

This papers introduces the first, to our knowledge, algorithm for the sparse recovery problem specifically designed for underdetermined partial circulant matrices. We give mathematical justifications for its derivation and validate our ideas empirically by some numerical results.

The remaining parts of the paper are organized as follows. We first summarize the major (numerical) linear algebra results that we use. The most important theorem involves the Fourier transform of a circulant matrix and its eigenvalues. Then, in Section III we review classical results from compressed sensing and in particular recovery guarantees for optimization and greedy methods. Then Section IV introduces our new recovery algorithm that is specifically designed for (random) partial circulant matrices. It first considers a noise free case to develop algorithm 1. We also provide ideas about how to deal with inaccurate measurements or lack of sparsity in the input unknown signal. Then Section V extends our findings to the case of doubly circulant block matrices. First empirical evidence of the well-foundedness of our algorithm are given in Section VI and we draw conclusions in Section VII.

II. Spectrum and inversion of circulant matrices

An $N \times N$ matrix $C = (c_{ij})_{i,j=1}^N$ is circulant if $c_{ij} = c_{i+1,j+1}$ and the subscripts are taken modulo N i.e.

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & \dots & c_{N-1} \\ c_{N-1} & c_0 & c_1 & \dots & c_{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & c_3 & \dots & c_0 \end{pmatrix} \tag{1}$$

and is denoted by $C = \mathrm{circ}(c_0, c_1, \dots, c_{N-1})$. C is diagonalizable by the Fourier matrix F: $C = F\Lambda F^*$, where $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_N)$ is the diagonal matrix of eigenvalues of C. Denote ω the N^{th} primitive root of unity, it holds

$$F(i,j) = \frac{1}{\sqrt{N}}\omega^{(i-1)(j-1)}$$

$$\Lambda_{(i,i)} = \lambda_i = \sum_{k=0}^{N-1} c_k(\omega^{i-1})^k \tag{2}$$

Once C is invertible,

$$C^{-1} = F\mathrm{diag}(\lambda_1^{-1}, \dots, \lambda_N^{-1})F^* \tag{3}$$

which implies that C^{-1} is a circulant matrix too. If C is singular, replace $(\lambda_1^{-1}, \dots, \lambda_N^{-1})$ by $(\lambda_1^{+}, \dots, \lambda_N^{+})$ in equation (3) to compute the Moore-Penrose pseudoinverse of C, denoted by C^+, where

$$\lambda_i^+ = \begin{cases} 0 & \text{if } \lambda_i = 0, \\ \lambda_i^{-1} & \text{if } \lambda_i \neq 0, \end{cases} \tag{4}$$

for $i = 1, 2, \dots, N$. Notice that $C^+ = C^{-1}$ when C is invertible. Then the l_2-minimization problem

$$\text{minimize } \|x\|_2, \text{ subject to } \mathrm{Cx} = \mathrm{b} \tag{5}$$

always has a solution C^+b (see [6], [7]).

III. Compressed sensing and sparse signal recovery

A. Compressed sensing

Compressed sensing (CS) has emerged in the last decade as a mathematical tool to solve underdetermined systems of linear equations $y = Ax \in \mathbb{R}^m$, for $x \in \mathbb{R}^N$, with $m \ll N$[1]. It relies on the assumption of sparsity of the solution (i.e. the number of non-zero components remains low), to find an adequate solution to the following problem:

$$\text{minimize } \|x\|_0, \quad \text{s.t. } \|Ax - y\|_2 \leq \eta\ , \tag{6}$$

where $\|x\|_0 = |\operatorname{supp}(x)|$ denotes the number of non-zero components (the sparsity), η is a constant handling the noise in the measurements. Eq. (6) defining an NP-hard mathematical program, it is usually replaced by its tightest convex relaxation, known as *Basis Pursuit DeNoising* (BPDN):

$$\text{minimize } \|x\|_1, \quad \text{s.t. } \|Ax - y\|_2 \leq \eta\ . \tag{7}$$

Usual conditions showing the equivalence of the two problems include [2] the *Null Space Property* (NSP) of the measurement matrix A up to a certain order; and the *Restricted Isometry Property* (RIP). A is said to satisfy the NSP of order s if, for any $S \subset \{1, \cdots, N\}$, $|S| = s$, and any vector $x \in \mathrm{Ker}(A)$ supported on S, it holds: $\|x_S\|_1 < \|x_{\overline{S}}\|_1$. A satisfies an RIP of order s and constant δ if for any s-sparse vector x, $|\|Ax\|_2^2 - \|x\|_2^2| \leq \delta\|x\|_2^2$. One can see this as a measure of closeness with an isometry on the set of s-sparse vectors. In particular, if $\delta_{2s} < 1$ then Problem (6) has a unique solution and for $\delta_{2s} < \sqrt{2} - 1$ the solution to Eq.. (7) is also the sparsest.

B. Algorithms and recovery guarantees

The different algorithms for sparse recovery can be classified in three main classes: algorithms based on optimization techniques such as linear programming and convex programming, greedy algorithms, such as *Orthogonal Matching Pursuit* (OMP) [8], and thresholding algorithms, such as *Compressive Sampling Matching Pursuit* (CoSaMP) [9], *Iterative Hard Thresholding* (IHT) [10], *Hard Thresholding Pursuit* (HTP) [11] and its variants *Graded HTP* (GHTP) [12], and *generalized HTP* (fHTP) [13]. These various algorithms have different characteristics, in particular regarding the number of non-zero entries in the sequences of estimates they yield; while HTP, CoSaMP, and IHT yield sequences of vectors of fixed sparsity, OMP, GHTP, and fHTP let the support grow with each iterations.

RIP-based recovery guarantees are usually expressed as sufficient conditions $\delta_{f(s)} < \overline{\delta}$, with $f(s)$ an integer valued function of the sparsity level s, and for an algorithm-dependent constant $\overline{\delta}$. Examples of such recovery conditions are given in Table I. The challenges reside both in finding appropriate

TABLE I

Usual sufficient conditions for exact sparse recovery from few measurements.

Alg	ℓ_1	OMP	HTP	GHTP	IHT	CoSaMP
$f(s)$	$2s$	$13s$	$3s$	$9s$	$3s$	$4s$
$\overline{\delta}$	$\sqrt{2} - 1$[2]	$1/6$	$1/\sqrt{3}$	$1/3$	0.5	$\frac{\sqrt{\sqrt{11/3}-1}}{2}$

matrices fulfilling such RIPs (note that it has been proven mainly for random matrices and partial circulant matrices) and also in increasing the range of convergence of such algorithms (i.e. increase the value $\overline{\delta}$, and reduce $f(s)$).

IV. PCMS: An algorithm to recover sparse signals from undersampled circulant measurements.

A partial circulant matrix Φ is a row submatrix of C with an arbitrary index set $\Omega = \{i_1, \cdots, i_m\} \subset \{1, 2, \dots, n\}$ whose cardinality $|\Omega| = m$ by removing all rows whose index is not in Ω from C (see [5], [15]). It will be assumed that the indices are ordered and unique: $i_1 < i_2 < \cdots < i_m$.

We describe here an algorithm designed for the particular structure of the matrix. It is divided into two main parts: a first optional part consist in recovering the set of indices of selected rows. This is not detailed in this note but very easy to derive and implement thank to the circulant structure of the matrix. The main part is a linear optimization problem in smaller dimensions (described in the following Subsection), instead of solving program (7). The two last sections are dedicated to the global description of the algorithm 1 as well as some inputs for how to deal with noisy observations.

A. Solving a smaller problem

Assuming that Ω can be easily recovered (more details will be given in the following section) we are left with $\Phi =_\Omega C$. Note that we will use the notation ${}_{\Omega_1}C_{\Omega_2}$ to denote the matrix extracted from C by keeping the rows indexed by Ω_1 and the columns indexed by Ω_2. Hence we are actually minimizing $\|C^{-1}b\|_1$ with $b = b_\Omega + b_{\overline{\Omega}}$ where b_Ω is the vector in $\mathbb{R}^N$ such that for any $1 \leq j \leq m$, $(b_\Omega)_{i_j} = y_j$ and 0 elsewhere.

[1]The complex case can also be considered, but we prefer to leave it aside for further research.

[2]This original bound has been later improved [14].

With $D := C^{-1}$ denoting the inverse of the circulant matrix C it holds

$$x = Db = D(b_\Omega + b_{\overline{\Omega}}) = D_\Omega y + D_{\overline{\Omega}} z \ , \tag{8}$$

for an unknown $z \in \mathbb{R}^{N-m}$. Hence the minimization problem Eq. (7) can be rewritten as

$$\text{minimize}_{z \in \mathbb{R}^{N-m}} \|D_{\overline{\Omega}} z + D_\Omega y\|_1. \tag{9}$$

This problem is actually equivalent to the usual Basis Pursuit (i.e. Program (7) in the noiseless case: $\eta = 0$).

Remark 1. It is now the right moment to clarify a few things. This method is clearly only applicable when the partial measurement matrix can be fully recreated. In this sense, this approach only applies to partial circulant matrices at the moment. Also it can only be applied to rather well posed circulant matrices. This is however not a main problem as the condition numbers of the random circulant matrices used here are rather well behaved [**?**], [16], [17]. It is worth mentioning that the idea introduced here could also be extended, in a certain way, to sub-matrices arising from invertible matrices.

Remark 2. Another point to notice is that, instead of minimizing over an N dimensional subspace, we are minimizing only on an $N - m$ dimensional subspace, hence the smaller computational costs. This is particularly important when dealing with larger sparsities. Indeed, stable recovery of most of the algorithms are ensured for a number of measurements m that scales with $s \log(N/s)$. As a consequence, the larger the sparsity, the more measurements we have and hence the smaller the search space.

Our approach shows advantages both in terms of computations and in terms of memory requirements. As the matrix C and its inverse are both circulant, only one vector of size N is needed. Moreover, the matrices being diagonalizable in a Fourier basis, all matrix-vector products can be carried via fast Fourier transforms.

B. Algorithmic concerns

The algorithm can be described only in a few steps:

Require: $\Phi \in \mathbb{R}^{m \times N}$, $y \in \mathbb{R}^m$,
return $x^\# \in \mathbb{R}^N$: the estimated solution to (9).
Find Ω (if not given)
Calculate the eigenvalues of the inverse of the full circulant matrix using Eq. (2) and Eq. (4).
Store the inverse eigenvalues.
Calculate $D_\Omega y$.
Define $D_{\overline{\Omega}}$.
Solve the convex problem (9).

Algorithm 1: Sparse signal recovery from partial circulant matrices

In particular, it is important to notice that, if the memory is a concern, then only saving the vector c (the first row of the partial circulant matrix Φ), the eigenvalues, and the vector $D_\Omega y$ is sufficient for the whole process. Moreover, all the matrices used here being (partial) circulant, the use of Fast Fourier Transform would drastically increase the speed.

C. Dealing with noisy observations

In a more realistic scenario the measurements will be corrupted by some additive noise as $y = \Phi x + e$, where we assume some bound $\|e\|_2 \leq \varepsilon$. The error might also stem from some lack of sparsity of the original vector x. It is often the case that the signal is only approximately sparse. In this scenario, the vector x can be decomposed on two sets S, containing the s largest entries in magnitude, and $\overline{S}$, where the entries on $\overline{S}$ are assumed to be negligible compared to the ones on S. Note that we can merge the two kind of noise into a single measurement noise vector $e' = e + \Phi x_{\overline{S}}$.

If we now reconsider the optimization problem described in Eq. (9), we see that the underlying assumption is the equality constraint $\Phi x = y$. We need to find constraints on the noise similar to the ones in Eq. (7) in order to tolerate for uncertainty in the measurements. Considering the inequality $\|\Phi x - y\|_2 \leq \eta$, together with Eq. (8), we get

$$\begin{aligned} & \|\Phi\left(D_\Omega y + D_{\overline{\Omega}} z\right) - y\|_2 \\ = & \ \|\left(\Phi D_\Omega - I\right) y + \Phi D_{\overline{\Omega}} z\|_2 \leq \eta \end{aligned} \tag{10}$$

The first part of this term is expected to be close to 0 as Φ and D_Ω are inverses of each other and the matrices are well posed. This approach, however, slightly modifies the objective function, but it must be noticed that $\|D_{\overline{\Omega}} z + D_\Omega y - x\|_1 = \|D_{\overline{\Omega}} z + D_\Omega e\|_1 \leq \|D_{\overline{\Omega}} z\|_1 + \|D_\Omega e\|_1$. This suggests that for small noise, minimizing $\|D_{\overline{\Omega}} z + D_\Omega y\|_1$ remains a good strategy. It is still to be investigated thoroughly.

V. Extensions

In this section, we extend our idea to solve the l_1-minimization problem (7) with a partial block circulant matrix. Let $N = sn$. An $N \times N$ block circulant matrix G is defined as $\text{circ}(G_0, G_1, \ldots, G_{s-1})$ where each block $G_j = \text{circ}(g_{j0}, \ldots, g_{j,n-1})$ is a circulant matrix. Denote $(\lambda_1^j, \lambda_2^j, \ldots, \lambda_n^j)$ the eigenvalues of G_j for $j = 0, 1, \ldots, s-1$. Denote F the $n \times n$ Fourier matrix and let

$$Q = \text{diag}(F, F, \ldots, F),$$

a $N \times N$ block Fourier matrix. Then $Q^* G Q = \text{circ}(D_1, D_2, \ldots, D_{s-1})$ where $D_j = \text{diag}(\lambda_1^j, \lambda_2^j, \ldots, \lambda_n^j)$ for $j = 0, 1, \ldots, s-1$. Then there exists an $N \times N$ permutation matrix P, such that

$$\begin{aligned} P^T Q^* G Q P &= P\text{circ}(D_1, D_2, \ldots, D_{s-1}) P^T \\ &= \text{diag}(H_1, H_2, \ldots, H_n) \end{aligned}$$

where

$$H_i = \text{circ}(\lambda_i^0, \lambda_i^1, \ldots, \lambda_i^{s-1})$$

for $i = 1, 2, \ldots, n$. The inverse of H_i can be computed by (3) if G_i's are invertible. This yields

$$G^{-1} = QP\text{diag}(H_1^{-1}, H_2^{-1}, \ldots, H_n^{-1}) P^T Q^*. \tag{11}$$

If the G_i's are singular, the inverse are replaced by the pseudo-inverse. With $L = QP\text{diag}(H_1^+, H_2^+, \ldots, H_n^+)P^T Q^*$, the l_2-minimization problem

$$\text{minimize} \|x\|_2, \text{ subject to Gx} = \text{b} \tag{12}$$

always has a solution Lb.

We define a partial block circulant matrix Ψ of G with an arbitrary set $\Omega = \{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, N\}$ whose cardinality $|\Omega| = m$ by removing all rows whose indices are not in Ω from G. The l_1-minimization problem

$$\text{minimize}_{x \in \mathbb{R}^N} \|x\|_1 \quad \text{subject to} \quad \Psi x = b, \tag{13}$$

can be reduced to a smaller l_1-minimization problem

$$\text{minimize}_{z \in \mathbb{R}^{N-m}} \|L_{\overline{\Omega}} z + L_\Omega y\|_1 \tag{14}$$

via the inverse of full block circulant matrix G containing Ψ. The algorithm can be described as the following:

Require: $\Phi \in \mathbb{R}^{m \times N}$, $y \in \mathbb{R}^m$, n

return $x^\# \in \mathbb{R}^N$: the estimated solution to (13).

Find Ω, the set of indices used for the row subselection.

Calculate the eigenvalues of the inverse of each circulant block using Eq. (2) and Eq. (4).

Store all the inverse eigenvalues.

Generate the permutation matrix P.

Calculate $L_\Omega y$.

Define $L_{\overline{\Omega}}$.

Solve the convex problem (14).

Algorithm 2: Sparse signal recovery from partial block circulant matrices

VI. Numerical validation

We describe now some first experiments validating the well-foundedness of the approach. As suggested in [5], we have used circulant matrices constructed from Gaussian sequences, as they fulfil a certain RIP. The support of the vectors to recover were generated at random with a fix number of non-zeros components s. The magnitude of the entries on the support were generated at random either from a Rademacher distribution or from a Gaussian distribution. The set Ω is generated by picking uniformly at random m elements in $\{1, \cdots, N\}$. For the optimization problems, we have chosen to use CVX, a package for specifying and solving convex programs [18], [19], but other packages such as ℓ_1-magic [20] may be of interests. All the numerical experiments are available for reproducible research from the author's webpage [3]. Note that smarter optimizations are possible for the original ℓ_1 problem, such as FISTA [21], but haven't been adapted to the particular approach introduced here. It is expected that similar optimization procedure would drasatically speed up the calculations in our novel algorithm.

We illustrate our results against some of the algorithms described in Section III, namely HTP and OMP. It is important to verify that our algorithm performs similarly to the usual ℓ_1 minimization (7) (Basis Pursuit). First, Fig. 1 shows the percentage of recovery over the random generation of 25 different matrices and vectors. Accurate recovery is understood in the relative error:

$$\frac{\|x - x^\#\|_2}{\|x\|_2} \leq \theta,$$

with θ set to 10^{-5}. Here we have $N = 400$ and the sparsity is fixed to 120. As hoped, the algorithm introduced in this note performs as good as the original ℓ_1 optimization, implemented in a similar fashion (again, better optimization toolboxes might disprove our claim).

In terms of computing time, we compare with the usual ℓ_1 basis pursuit optimization. We are aware that faster solvers are available, but comparison with them would not be fair until a better optimization is derived for the case of partial circulant matrices.

TABLE II
Average time, in seconds, for exact recovery. $N = 400$, $s = 120$, $m = 300$.

Sensing matrix	Gaussian		Rademacher	
Vector	Gaussian	Uniform	Gaussian	Uniform
PCM	1.5783	1.8746	1.6308	1.8646
ℓ_1	4.2440	4.6092	3.9561	4.5883

Table II compares the computing time for the original basis pursuit and our novel approach for four use cases: Gaussian and Rademacher circulant sensing matrices, together with Gaussian and uniform unknown vectors. Only the times for perfect recovery were used for the average. As can be seen, in all of the situations, the PCM method performs much faster. Further numerical results, not reported here, suggest that the speed improvement is greater as the system gets bigger.

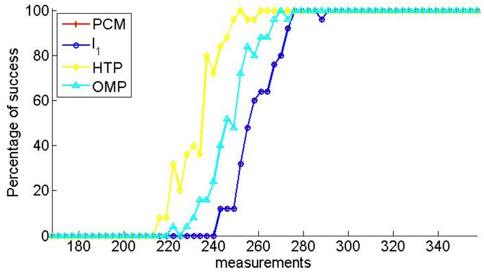
Finally, a last set of experiments, illustrated in Fig. 2, shows the recovery capabilities of our algorithm in presence of additive noise. For these experiments, the recovery threshold θ is set to 10^{-2} and the noise stems from a Gaussian distribution with variance 10^{-5}. As can be seen, the performance of the algorithm does not degrade when facing little noise. Further analysis needs to be done to investigate the impact of stronger noise.

VII. Conclusion

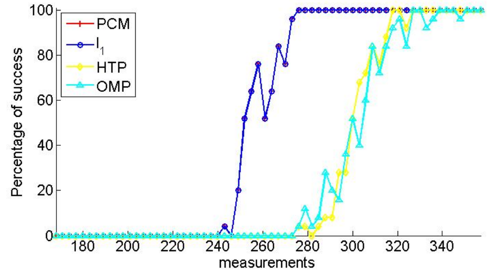
This note introduced the first bricks for an algorithm specifically designed for the recovery of sparse signals from few measurement when these measurement stem from a circulant matrix. We derived a mathematical formulation based on a ℓ_1 minimization problem which makes use of the particular structure of the matrix.

The approach is easy to implement, fast, and memory efficient. By its derivation based on the diagonalization of the sensing matrix via Fourier basis, we are able to reconstruct the full set of measurements virtually and simulate a classic matrix inversion problem. Though this inversion still need

[3] See http://www.mathc.rwth-aachen.de/en/~bouchot/home

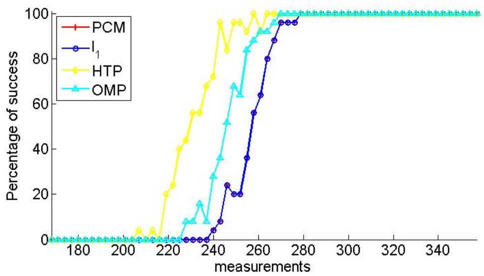


(a) Gaussian partial cirulant matrix, Gaussian signal

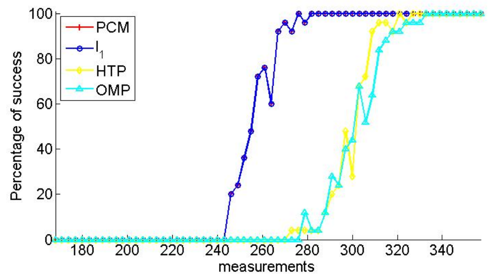


(b) Gaussian partial cirulant matrix, Rademacher signal

Fig. 1. Percentage of recovery in various scenarios for different recovery algorithms (noiseless).



(a) Gaussian partial cirulant matrix, Gaussian signal



(b) Gaussian partial cirulant matrix, Rademacher signal

Fig. 2. Percentage of recovery in various scenarios for different recovery algorithms (with additive noise).

better processing, first experiments suggest that this method is reliable.

The first ideas introduced here still require a thorough analysis of the method, in order to theoretically justify its speed and efficacy. The encouraging numerical results motivate for further research and understanding.

Acknowledgment

J.-L. B. is funded by the European Research Council through the Starting Grant StG 258926 (SPALORA).

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