Characterization of Besov spaces on nested fractals by piecewise harmonic functions

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Abstract

In the present paper we characterize the Besov spaces $B^{s}_{pq}(\Gamma, \mu)$ on nested fractals in terms of the coefficients of functions with respect to the piecewise harmonic basis.

Key words: Besov spaces, traces, nested fractals

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1 Introduction

Besov spaces $B^{s}_{pq}(\Gamma, \mu)$ on $d$-sets in $\mathbb{R}^n$ can be defined by traces of Besov spaces $B^{s+\frac{2-d}{p}}_{pq}(\mathbb{R}^n)$. When the smoothness parameter $s$ is small, $B^{s}_{pq}(\Gamma, \mu)$ can be characterized by intrinsic building blocks, namely atoms. In the present paper first we give a characterization of Besov spaces by new type of atoms, which we call $(s,p,\sigma)$-atoms.

On the other hand Besov spaces on the most trivial example of a $d$-set, the unit interval, can be described by means of Faber-Schauder basis [18]. We are looking for its counterpart for the $d$-set $\Gamma$. So we need to find the description of functions in Faber-Schauder basis in such a way that it can be transferred to other sets. Our approach is to start with a Dirichlet form $(\mathcal{E}, \mathcal{D})$, see e.g. [8, 15]. Then the harmonic function on $\Gamma$ with given boundary
values can be defined as the unique function that minimizes $E(f)$. Similarly we can define piecewise harmonic functions. Piecewise harmonic functions on the unit interval are exactly the functions forming the Faber-Schauder basis. Thus the family of piecewise harmonic functions may be regarded as the counterpart of Faber-Schauder basis. Piecewise harmonic functions are Lipschitz with respect to the effective resistance metric. We additionally assume that $\Gamma$ is a nested fractal. Then the effective resistance metric is equivalent to the Euclidean metric taken to some power and this enables us to treat piecewise harmonic functions as $(s,p,\sigma)$-atoms. Thus functions from $B_{pq}^s(\Gamma,\mu)$ can be characterized in terms of the coefficients of its expansion in a piecewise harmonic basis.

The main result of the paper is contained in the Theorem 5. Our proof is based on the atomic characterization of Besov spaces. A similar result is also presented in the paper [13], where the harmonic representation of Lipschitz spaces $(\Lambda^{p,q}_\alpha(1))$ introduced by Strichartz is stated. It was shown in [1] that $(\Lambda^{p,q}_\alpha(1)) \Gamma$ coincide with $\text{Lip}(\alpha/\alpha_0,p,q,\Gamma)$, when $\Gamma$ is a nested fractal. Thus the harmonic representation of Besov spaces might be also proved by using the discrete characterizations of Besov spaces.

2 Preliminaries

2.1 Basic notation and classical Besov spaces

Let $\mathbb{N}$ be the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. $\mathbb{Z}$ is the set of all integers. Let $\mathbb{R}^n$ be Euclidean $n$-space, where $n \in \mathbb{N}$. The scalar product of $x, y \in \mathbb{R}^n$ is given by $xy = \sum_{i=1}^n x_i y_i$. Put $\mathbb{R} = \mathbb{R}^1$, whereas $\mathbb{C}$ is the complex plane. Let $S(\mathbb{R}^n)$ be the Schwartz space of all complex-valued, rapidly decreasing, infinitely differentiable functions on $\mathbb{R}^n$. By $S'(\mathbb{R}^n)$ we denote its topological dual, the space of all tempered distributions on $\mathbb{R}^n$. $L_p(\mathbb{R}^n)$ with $0 < p \leq \infty$, is the standard quasi-Banach space with respect to Lebesgue measure, quasi-normed by

$$\| f \|_{L_p(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{\frac{1}{p}}, \quad 0 < p < \infty,$$

$$\| f \|_{L_\infty(\mathbb{R}^n)} = \text{ess-sup}_{x \in \mathbb{R}^n} |f(x)|.$$
If $\varphi \in S(\mathbb{R}^n)$ then
\[
\hat{\varphi}(\xi) = \mathcal{F}\varphi(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \varphi(x)e^{-ix\xi} \, dx, \quad \xi \in \mathbb{R}^n,
\]
denotes the Fourier transform of $\varphi$. The inverse Fourier transform is given by
\[
\varphi^\vee(x) = \mathcal{F}^{-1}\varphi(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \varphi(\xi)e^{ix\xi} \, d\xi, \quad x \in \mathbb{R}^n.
\]
One extends $\mathcal{F}$ and $\mathcal{F}^{-1}$ in the usual way from $S$ to $S'$. For $f \in S'(\mathbb{R}^n)$,
\[
\mathcal{F}f(\varphi) = f(\mathcal{F}\varphi), \quad \varphi \in S(\mathbb{R}^n).
\]
Let $\varphi_0 \in S(\mathbb{R}^n)$ with
\[
\varphi_0(x) = 1, \quad |x| \leq 1 \quad \text{and} \quad \varphi_0(x) = 0, \quad |x| \geq \frac{3}{2}, \tag{1}
\]
and let
\[
\varphi_k(x) = \varphi_0(2^{-k}x) - \varphi_0(2^{-k+1}x), \quad x \in \mathbb{R}^n, \quad k \in \mathbb{N}. \tag{2}
\]
Then, since
\[
1 = \sum_{j=0}^{\infty} \varphi_j(x) \quad \text{for all} \quad x \in \mathbb{R}^n, \tag{3}
\]
the $\varphi_j$ form a dyadic resolution of unity in $\mathbb{R}^n$. According to the Paley-Wiener-Schwartz theorem $\left(\varphi_k \hat{f}\right)^\vee$ is an entire analytic function on $\mathbb{R}^n$ for any $f \in S'(\mathbb{R}^n)$. In particular, $\left(\varphi_k \hat{f}\right)^\vee(x)$ makes sense pointwise.

**Definition 1.** Let $\varphi = \{\varphi_j\}_{j=0}^{\infty}$ be the dyadic resolution of unity according to (1)-(3), $s \in \mathbb{R}$, $0 < p \leq \infty$, $0 < q \leq \infty$ and
\[
\|f\|_{B_{pq}^s(\mathbb{R}^n)} = \left(\sum_{j=0}^{\infty} 2^{jsq} \|\left(\varphi_k \hat{f}\right)^\vee\|_{L_p(\mathbb{R}^n)}^q\right)^{\frac{1}{q}}
\]
(with the usual modification if $q = \infty$). Then the Besov space $B_{pq}^s(\mathbb{R}^n)$ consists of all $f \in S'(\mathbb{R}^n)$ such that $\|f\|_{B_{pq}^s(\mathbb{R}^n)} < \infty$. 

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2.2 Trace spaces $B_{pq}^s(\Gamma, \mu)$

Definition 2. A compact set $\Gamma$ in $\mathbb{R}^n$ is called a $d$-set with $0 < d < n$ if there is a Radon measure $\mu$ in $\mathbb{R}^n$ with support $\Gamma$ such that for some positive constants $c_1$ and $c_2$, holds

$$c_1 r^d \leq \mu(B(\gamma, r)) \leq c_2 r^d, \quad \gamma \in \Gamma, \ 0 < r < 1. \quad (4)$$

where $B(x, r)$ is a ball in $\mathbb{R}^n$ centered at $x \in \mathbb{R}^n$ and of radius $r > 0$. The measure $\mu$ satisfying (4) is called a $d$-measure.

If $\Gamma$ is a $d$-set, then the restriction to $\Gamma$ of the $d$-dimensional Hausdorff measure $H^d$ satisfies (4) and any measure $\mu$ satisfying (4) is equivalent to $H^d|_\Gamma$.

Definition 3. Let $\mu$ be a Radon measure in $\mathbb{R}^n$. Let

$$s > 0, \ 1 < p < \infty, \ 0 < q < \infty. \quad (5)$$

Let for some $c > 0$,

$$\int_{\Gamma} |\varphi(\gamma)| \mu(d\gamma) \leq c \|\varphi|_{B_{pq}^s(\mathbb{R}^n)}\| \text{ for all } \varphi \in S(\mathbb{R}^n). \quad (6)$$

Then the trace operator $\text{tr}_\mu$,

$$\text{tr}_\mu : B_{pq}^s(\mathbb{R}^n) \hookrightarrow L_1(\Gamma, \mu),$$

is the completion of the pointwise trace $(\text{tr}_\mu \varphi)(\gamma) = \varphi(\gamma), \ \varphi \in S(\mathbb{R}^n)$. Furthermore, the image of $\text{tr}_\mu$ is denoted by $\text{tr}_\mu B_{pq}^s(\mathbb{R}^n)$ and is quasi-normed by

$$\|g|_{\text{tr}_\mu B_{pq}^s(\mathbb{R}^n)}\| = \inf \left\{ \|f|_{B_{pq}^s(\mathbb{R}^n)}\| : \text{tr}_\mu f = g \right\}.$$

Remark 1. The above definition is justified since $S(\mathbb{R}^n)$ is dense in $B_{pq}^s(\mathbb{R}^n)$ with (5). We refer to [16], Theorem 2.3.3, p. 48. Due to (6), the trace of $f$ is independent of the approximation of $f$ in $B_{pq}^s(\mathbb{R}^n)$ by $S(\mathbb{R}^n)$-functions.

Definition 4. Let $\Gamma$ be a $d$-set in $\mathbb{R}^n$. Let $s > 0, \ 1 < p < \infty, \ 0 < q < \infty$. Then

$$B_{pq}^s(\Gamma, \mu) = \text{tr}_\mu B_{pq}^{s+\frac{n-d}{p}}(\mathbb{R}^n)$$

with

$$\|f|_{B_{pq}^s(\Gamma, \mu)}\| = \inf \left\{ \|g|_{B_{pq}^s(\mathbb{R}^n)}\| : \text{tr}_\mu g = f \right\}.$$

There is the extension operator which is closely connected to the trace operator. The following assertion is covered by the Theorem 3, p.155 in [6], we also refer to [17, Section 1.17.2].

**Theorem 1.** Let $\Gamma$ be a compact $d$-set in $\mathbb{R}^n$ with $0 < d < n$ and let $\mu$ be a corresponding Radon measure. Let

$$0 < s < 1, \quad 1 < p < \infty, \quad 1 \leq q \leq \infty, \quad t = s + (n - d)/p,$$

and let $\text{tr}_\mu$ be the trace operator. Then there is a linear and bounded extension operator $\text{ext}_\mu$ with

$$\text{ext}_\mu : B^s_{pq}(\Gamma, \mu) \hookrightarrow B^t_{pq}(\mathbb{R}^n)$$

and

$$\text{tr}_\mu \circ \text{ext}_\mu = \text{id} \quad \text{(identity in } B^s_{pq}(\Gamma, \mu)).$$

All our reasoning strongly uses the fact that $B^s_{pp}(\Gamma)$ with $1 < p < \infty$ and $0 < s < 1$ can be equivalently normed by

$$\|f\|_{B^s_{pp}(\Gamma)} = \left( \int_{\Gamma} |f(\gamma)|^p \mu(d\gamma) + \int_{\Gamma} \int_{\Gamma} \frac{|f(\gamma) - f(\delta)|^p}{|\gamma - \delta|^{d+sp}} \mu(d\gamma)\mu(d\delta) \right)^{1/p},$$

we refer to [6]. We simplify the notation and write $B^s_p(\Gamma)$ instead of $B^s_{pp}(\Gamma, \mu)$.

### 3 Atomic characterizations of $B^s_p(\Gamma)$

Besov spaces $B^s_p(\Gamma)$ with $0 < s < 1$ and $1 < p < \infty$ can be characterized in terms of intrinsic building blocks, namely atoms.

Let for $\delta > 0$

$$\Gamma_\delta = \bigcup_{\gamma \in \Gamma} B(\gamma, \delta),$$

where

$$B(\gamma, \delta) = \{x \in \mathbb{R}^n : |x - \gamma| < \delta\},$$

be a $\delta$-neighbourhood of $\Gamma$. Let $0 < r < 1$ be fixed. Let for $j \in \mathbb{N}_0$,

$$\{\gamma_{j,m}\}_{m=1}^{M_j} \subset \Gamma$$

be the lattice of points with the following properties:
• For some $c_1 > 0$

\[ |\gamma_{j,m_1} - \gamma_{j,m_2}| \geq c_1 r^j, \ j \in \mathbb{N}_0, \ m_1 \neq m_2. \] (12)

• For some $c_2 > 0$

\[ \Gamma_{c_2 r^j} \subset \bigcup_{m=1}^{M_j} B(\gamma_{j,m}, r^j), \ j \in \mathbb{N}_0, \] (13)

where $B(\gamma_{j,m}, r^j)$ are given by (10).

Let

\[ B^r_{j,m} = \{ \gamma \in \Gamma : |\gamma - \gamma_{j,m}| < r^j \}, \ j \in \mathbb{N}_0, \ m = 1, \ldots, M_j, \] (14)

be the intersection of balls $B(\gamma_{j,m}, r^j)$ with $\Gamma$.

**Definition 5.** Let $\Gamma$ be a $d$-set in $\mathbb{R}^n$. Let $1 < p < \infty$ and $0 < s < 1$. Then a continuous function $a_{jm}$ on $\Gamma$ is called an $(s,p)$-atom, if for $j \in \mathbb{N}_0$ and $m = 1, \ldots, M_j$,

\[ \text{supp} a_{jm} \subset B^r_{j,m}, \] (15)

\[ |a_{jm}(\gamma)| \leq H^d \left( B^r_{j,m} \right)^{\frac{d-1}{p}}, \ \gamma \in \Gamma, \] (16)

and

\[ |a_{jm}(\gamma) - a_{jm}(\delta)| \leq H^d \left( B^r_{j,m} \right)^{\frac{d-1}{p}} |\gamma - \delta| \] (17)

with $\gamma, \delta \in \Gamma$, [17, Section 8.1.3].

Since $\Gamma$ is a $d$-set, we can reformulate (16) and (17) as

\[ |a_{jm}(\gamma)| \leq cr^{j(s-\frac{d}{p})}, \]

\[ |a_{jm}(\gamma) - a_{jm}(\delta)| \leq cr^{j(s-1-\frac{d}{p})} |\gamma - \delta|. \]

For our further purposes we need the following assertion which is covered by the Proposition 8.10 in [17].

**Lemma 1.** Let $\Gamma$ be a $d$-set. Let $r \geq 0$ and

\[ B^\Gamma(r) = \{ \gamma \in \Gamma : |\gamma - \gamma_0| < r \} \text{ for some } \gamma_0 \in \Gamma, \]

and

\[ B(2r) = \{ x \in \mathbb{R}^n : |x - \gamma_0| < 2r \}. \]
Let 
\[ f \in B^s_p(\Gamma) \text{ with } \text{supp} f \subset B^r(\Gamma). \]

Then 
\[ \|f|B^s_p(\Gamma)\| = \inf \|g|B^t_p(\mathbb{R}^n)\|, \quad t = s + (n - d)/p, \]
where the infimum is taken over all 
\[ g \in B^t_p(\mathbb{R}^n), \quad g|_{\Gamma} = f, \quad \text{supp} g \subset B(2r). \]

Now we can formulate an intrinsic atomic decomposition of the trace spaces \( B^s_p(\Gamma) \).

**Theorem 2.** Let \( 1 < p < \infty, \) and \( 0 < s < 1. \) Then \( B^s_p(\Gamma) \) is the collection of all \( f \in L^1(\Gamma, \mu) \) which can be represented as

\[ f(\gamma) = \sum_{j=0}^{\infty} \sum_{m=1}^{M_j} \lambda^j_m a_{jm}(\gamma), \quad \gamma \in \Gamma, \quad (18) \]

where

\[ \|\lambda\| = \left( \sum_{j=0}^{\infty} \sum_{m=1}^{M_j} |\lambda^j_m|^p \right)^{\frac{1}{p}} < \infty, \]

\( a_{jm} \) are \((s, p)^*\)-atoms according to Definition 5 and (18) converges absolutely in \( L^1(\Gamma, \mu) \). Furthermore,

\[ \|f|B^s_p(\Gamma)\| \sim \inf \|\lambda\| \quad (19) \]

where infimum is taken over all admissible representations (18), [17, Chapter 8.1.3].

We introduce new type of atoms, that we call \((s, p, \sigma)\)-atoms.

**Definition 6.** Let \( 1 < p < \infty, \ 0 < \sigma < 1 \) and \( 0 < s < \sigma. \) Then a continuous function \( a_{jm} \) on \( \Gamma \) is called an \((s, p, \sigma)\)-atom, if for \( j \in \mathbb{N}_0 \) and \( m = 1, \ldots, M_j, \)

\[ \text{supp} a_{jm} \subset B^r_{j,m}, \quad (20) \]

\[ |a_{jm}(\gamma)| \leq cr^j(s-\frac{\sigma}{p}), \quad \gamma \in \Gamma, \quad (21) \]

and

\[ |a_{jm}(\gamma) - a_{jm}(\delta)| \leq cr^j(s-\sigma-\frac{\sigma}{p}) |\gamma - \delta|^{\sigma} \quad (22) \]

with \( \gamma, \delta \in \Gamma. \)
Let $a_{jm}$ be an $(s,p)$-atom and $0 < s < \sigma$. Then
\[
|a_{jm}(\gamma) - a_{jm}(\delta)| \leq cr_j^{(s-1-\frac{d}{p})} |\gamma - \delta|
\]
\[
= cr_j^{(s-1-\frac{d}{p})} |\gamma - \delta|^{1-\sigma} |\gamma - \delta|^\sigma \leq cr_j^{(s-1-\frac{d}{p})} r_j^{(1-\sigma)} |\gamma - \delta|^\sigma
\]
\[
= cr_j^{(s-\sigma-\frac{d}{p})} |\gamma - \delta|^\sigma,
\]
which shows that any $(s,p)$-atom is an $(s,p,\sigma)$-atom.

**Theorem 3.** Let $1 < p < \infty$, $0 < \sigma < 1$ and $0 < s < \sigma$. Then $B^*_p(\Gamma)$ is the collection of all $f \in L_1(\Gamma, \mu)$ which can be represented as
\[
f(\gamma) = \sum_{j=0}^{\infty} \sum_{m=1}^{M_j} \lambda_{jm}^\gamma a_{jm}(\gamma), \ \gamma \in \Gamma,
\]
where
\[
||\lambda|| = \left( \sum_{j=0}^{\infty} \sum_{m=1}^{M_j} |\lambda_{jm}|^p \right)^{\frac{1}{p}} < \infty,
\]
a_{jm} are $(s,p,\sigma)$-atoms according to Definition 6 and (23) converges absolutely in $L_1(\Gamma, \mu)$. Furthermore,
\[
||f|B^*_p(\Gamma)|| \sim \inf ||\lambda||
\]
where infimum is taken over all admissible representations (23).

**Proof.** The proof is the adaption of reasoning in [17, Section 8.1.3]. The representation (18) with $(s,p)^*$-atoms is a special case of the representation (23) and it holds (19). Hence it remains to show that from the representation (23) follows that
\[
f \in B^*_p(\Gamma) \text{ and } ||f|B^*_p(\Gamma)|| \leq c||\lambda||.
\]

First we estimate the norm of $(s,p,\sigma)$-atoms in $B^*_p(\Gamma)$. Let $L$ be a number such that $\text{diam} \Gamma \leq 2^L$. Then
\[
\int \int \frac{|\lambda_{jm}(\gamma) - \lambda_{jm}(\delta)|^p}{|\gamma - \delta|^{d+sp}} \mu(d\delta) \mu(d\gamma) \leq c \int \int \frac{1}{|\gamma - \delta|^{d+(s-\sigma)p}} \mu(d\delta) \mu(d\gamma)
\]
\[ = c \int_{\Gamma} \sum_{i=-\infty}^{L} \int_{B(\gamma,2i) \setminus B(\gamma,2i-1)} \frac{1}{|\gamma - \delta|^{d+(s-\sigma)p}} \mu(d\delta) \mu(d\gamma) \]
\[ \leq c \int_{\Gamma} \sum_{i=-\infty}^{L} \int_{B(\gamma,2i) \setminus B(\gamma,2i-1)} \frac{1}{2i(d+(s-\sigma)p)} \mu(d\delta) \mu(d\gamma) \]
\[ \leq c\mu(\Gamma) \sum_{i=-\infty}^{L} \frac{2^{di}}{2^{i(d+(s-\sigma)p)}} = c\mu(\Gamma) \frac{2^{L(s-\sigma)p}}{1 - 2^{(s-\sigma)p}} \leq C. \]

Moreover,
\[ \int_{\Gamma} |a_{jm}(\gamma)|^p \mu(d\gamma) \leq \int_{B_{jm}} \mu(B_{jm}) \frac{\mu(d\gamma)}{\mu(B_{jm})} \leq \mu(\Gamma) \frac{\mu(d\gamma)}{\mu(B_{jm})} = C. \]

This means that there is a constant \( C > 0 \) such that
\[ \|a_{jm}|B_{\sigma}^p(\Gamma)\| \leq C \]
for all \((s,p,\sigma)\)-atoms. Furthermore, for \( 0 < s \leq \bar{s} < \sigma \) we can write
\[ a_{jm}(\gamma) = r^{j(s-\bar{s})} b_{jm}(\gamma), \]
where
\[ b_{jm}(\gamma) = r^{j(s-\bar{s})} a_{jm}(\gamma). \]

For each \( j \in \mathbb{N}_0 \) and \( m = 1, \ldots, M_j \) we have
\[ \text{supp } b_{jm} = \text{supp } a_{jm} \subset B_{jm}^F, \]
\[ |b_{jm}(\gamma)| \leq cr^{j(\bar{s}-\frac{d}{p})} \]
and
\[ |b_{jm}(\gamma) - b_{jm}(\delta)| \leq cr^{j(\bar{s}-\sigma-\frac{d}{p})} |\gamma - \delta|^{\sigma}. \]

This shows that \( b_{jm} \) are \((\bar{s},p,\sigma)\)-atoms and
\[ \|b_{jm}|B_{p}^\bar{s}(\Gamma)\| \leq C. \]
Hence
\[ \|a_{jm}|B_{p}^\bar{s}(\Gamma)\| \leq C r^{j(s-\bar{s})}. \]
We apply Lemma 1 to $a_{jm}$. Then it follows that there are functions $A_{jm} \in B_{pp}^{\tilde{t}}(\mathbb{R}^n)$, where $\tilde{t} = \bar{s} + \frac{n - d}{p}$, such that

$$\text{tr}_\mu A_{jm} = a_{jm}, \quad \text{supp} A_{jm} \subset \{ x \in \mathbb{R}^n : |x - \gamma_{jm}| \leq c_1 r^j \}$$

and

$$\| A_{jm} \|_{B_{pp}^{\tilde{t}}(\mathbb{R}^n)} \leq c_2 r^j (t - \bar{t}), \quad t = s + \frac{n - d}{p}.$$ 

Then according to the Definition 2.7 in [17] $A_{jm}$ are non-smooth atoms for $B_{pp}^{t}(\mathbb{R}^n)$ and from Theorem 2.3 in [17] follows that

$$F = \sum_{j=0}^{\infty} \sum_{m=1}^{M_j} \lambda_{jm} A_{jm} \quad \text{with} \quad \| \lambda \| < \infty$$

belongs to $B_{pp}^{t}(\mathbb{R}^n)$ and

$$\| F \|_{B_{pp}^{t}(\mathbb{R}^n)} \leq c \| \lambda \|.$$

Taking into account that $f = \text{tr}_\mu F$, we may conclude

$$\| f \|_{B_p^{s}(\Gamma)} \leq c \| \lambda \|.$$

\[\Box\]

4 Self-similar sets

Typical examples of $d$-sets are self-similar sets with invariant measure $\mu$. Generally speaking, a self-similar set is a set that is made up of parts which are similar to the whole. The mathematical definition was given by Hutchinson in [4].

\textbf{Definition 7.} A mapping $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called a similarity (similitude), if there is a constant $0 < \rho < 1$ such that for all $x, y \in \mathbb{R}^n$ holds

$$|F(x) - F(y)| = \rho \left| x - y \right|.$$ 

The constant $\rho$ is called the contraction ratio of $F$. 

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Theorem 4. Let \( \{F_i\}_{i=1}^N \) be similarities in \( \mathbb{R}^n \). Then there exists a unique non-empty compact set \( \Gamma \subset \mathbb{R}^n \) that satisfies

\[
\Gamma = \bigcup_{i=1}^N F_i(\Gamma).
\] (24)

\( \Gamma \) is called a self-similar set with respect to \( \{F_i\}_{i=1}^N \).

There are many books and papers dealing with self-similar sets, we refer to [2], [4] and [8].

A set \( \Gamma_w \) with \( w = (w_1, w_2, \ldots, w_j) \), \( w_i \in \{1, \ldots, N\} \) defined by

\[
\Gamma_w = F_w(\Gamma) = F_{w_1} \circ F_{w_2} \circ \ldots \circ F_{w_j}(\Gamma),
\]
is called \( j \)-simplex. We call \( w \) a word of length \( j = |w| \). Then holds

\[
\Gamma = \bigcup_{|w|=j} F_w(\Gamma).
\] (25)

Let \( \Sigma \) be a set of all infinite sequences

\[
\Sigma = \{(\omega_1, \omega_2, \ldots) : \omega_i \in \{1, 2, \ldots, N\}\}.
\]

For any \( \omega = (\omega_1, \omega_2, \ldots) \in \Sigma \) define a continuous surjective map \( \pi : \Sigma \to \Gamma \) by

\[
\pi(\omega) = \bigcap_{m=1}^\infty \Gamma_{\omega_1 \omega_2 \ldots \omega_m}.
\]

Let

\[
C = \bigcup_{i \neq j} (\Gamma_i \cap \Gamma_j),
\]

\( \mathcal{C} = \pi^{-1}(C) \) and \( \mathcal{P} = \bigcup_{n \geq 1} \sigma^n(\mathcal{C}) \),

where \( \sigma : \Sigma \to \Sigma \) is the shift map defined by

\[
\sigma(\omega_1, \omega_2, \ldots) = (\omega_2, \omega_3, \ldots).
\]

Let

\[
V_0 = \pi(\mathcal{P}) \text{ and } V_j = \bigcup_{i=1}^N F_i(V_{j-1}),
\]

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or equivalently

\[ V_j = \bigcup_{|w|=j} F_w(V_0). \]

Then \( V_j \) describes the set of boundary points of simplexes of fixed level \( j \). It is clear that \( V_j \subset V_{j+1} \).

Let \( V_s = \bigcup_{j=0}^{\infty} V_j \), then \( \Gamma = V_s \) in the Euclidean topology. When \( j \) is fixed, \( V_j \) is the natural lattice of points in the self-similar set \( \Gamma \) that satisfies (12) and (13). We followed [8, Sections 1.2-1.3]. We form a graph \( G_j \) with vertices \( V_j \) and edge relation \( \xi \sim_j \eta \) holding if and only if there exists a \( j \)-simplex containing both \( \xi \) and \( \eta \) as boundary points.

In this paper we consider sets \( \Gamma \) such that they are self-similar with respect to the similarities with the same contraction ratio \( 0 < \rho < 1 \), that is

\[ |F_i(x) - F_i(y)| = \rho |x - y|. \] (26)

The unit interval \( I = [0, 1] \) is a self-similar set with respect to the similarities \( F_i : \mathbb{R} \to \mathbb{R}, i = 1, 2, \)

\[ F_1(x) = \frac{1}{2}x, \quad F_2(x) = \frac{1}{2}x + \frac{1}{2}. \]

The Koch curve \( K \) is a self-similar set with respect to the similarities \( F_i : \mathbb{R}^2 \to \mathbb{R}^2, i = 1, 2, \)

\[ F_1(x, y) = \left( \frac{1}{2}x + \frac{1}{2\sqrt{3}}y, \frac{1}{2\sqrt{3}}x - \frac{1}{2}y \right), \]
\[ F_2(x, y) = \left( \frac{1}{2}x - \frac{1}{2\sqrt{3}}y + \frac{1}{2}, -\frac{1}{2\sqrt{3}}x - \frac{1}{2}y + \frac{1}{2\sqrt{3}} \right), \]

see [8], where mappings \( F_1, F_2 \) are given in a complex form. The self-similar structure of the unit interval \( I \) and the Koch curve \( K \) can be used to establish the transform

\[ H : I \to K, \] (27)

such that

\[ |H(x) - H(y)|^d \sim |x - y|, \] (28)
where $d = \frac{\ln 4}{\ln 3}$ is the Hausdorff dimension of the Koch curve $K$. For the analytical expression of $H$ we refer to [8, Example 1.2.7], some information can be also find in [17, Section 8.2.2].

There is a special kind of sets that are self-similar with respect to similarities (26), satisfying some additional properties, known as nested fractals. They were first introduced by Lindstrøm [11], and afterwards were studied by many authors, e.g. [10, 12]. Nested fractals should satisfy following conditions:

**C0.** $\#V_0 \geq 2$.

**C1. Open set condition**
The family of similarities $\{F_i\}_{i=1}^N$ satisfies the open set condition if there exists an open, bounded, nonempty set $O \subset \mathbb{R}^n$ such that

$$F_i(O) \cap F_j(O) = \emptyset \text{ for } i \neq j$$

and

$$\bigcup_{i=1}^N F_i(O) \subset O.$$  

When the open set condition is satisfied, the Hausdorff dimension $d$ of $\Gamma$ is

$$d = \frac{\log N}{\log \frac{1}{\rho}},$$

we refer to [2, 4].

**C2. Nesting**
If $j \geq 1$ and $w = (w_1, w_2, \ldots, w_j)$ and $w' = (w'_1, w'_2, \ldots, w'_j)$ are distinct elements of $\{1, 2, \ldots, N\}^n$, then

$$\Gamma_w \cap \Gamma_{w'} = F_w(V_0) \cap F_{w'}(V_0).$$

**C3. Connectivity**
The graph $(V_1, G_1)$ is connected.

**C4. Symmetry**
For any $x, y \in \mathbb{R}^n$ with $x \neq y$, let $H_{xy}$ denote the hyperplane given by

$$H_{xy} = \{z \in \mathbb{R}^n : |z - x| = |z - y|\}$$
and let $R_{xy}$ denote the reflection with respect to $H_{xy}$. Then for any $x, y \in V_0$ with $x \neq y$, $R_{xy}$ maps $j$-cells to $j$-cells, and maps any $j$-cell which contains elements in both sides of $H_{xy}$ to itself for each $j \geq 0$.

The simplest example of the nested fractal is the Sierpinski gasket $SG$, which is generated by three similarities in the plane $F_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $i = 1, 2, 3$, defined by

$$F_i(x) = \frac{1}{2}(x - \xi_i) + \xi_i,$$ (29)

where $\xi_i$ are the vertices of an equilateral triangle, see [15, Section 1.1].

Further on we assume that the diameter of $\Gamma$ is 1. Then the diameter of each $j$-simplex $\Gamma_{w_1...w_j}$ is $\rho^j$, where $\rho$ is from (26). In case of $I$ and $SG$ we get $\rho = \frac{1}{2}$, in case of $K$ we have $\rho = \frac{1}{\sqrt{3}}$.

Suppose a real-valued function $u$ is given on the vertices $V_j$. Then there is a natural Dirichlet form

$$E_j(u) = \sum_{\xi \sim \eta} (u(\xi) - u(\eta))^2.$$ We need to multiply $E_j$ by the renormalization factor $\alpha^j$ in order to have the following consistency property:

**Lemma 2.** For every function $u$ on $V_j$ there exists a unique extension $\tilde{u}$ to $V_{j+1}$ minimizing $E_{j+1}$, i.e.

$$E_{j+1}(\tilde{u}) = \min \left\{ E_{j+1}(u') : u'|_{V_j} = u \right\},$$

and

$$\alpha^j E_j(u) = \alpha^{j+1} E_{j+1}(\tilde{u}).$$ (30)

For $I$ and $K$ the renormalization factor $\alpha$ is equal to 2, for $SG$ we have $\alpha = \frac{2}{3}$, [15, Section 1.3]. The number $d_w = \frac{\log N\alpha}{\log \frac{1}{\rho}}$ is called the walk dimension of $\Gamma$. The renormalized graph energies are defined by

$$\mathcal{E}_j(u) = \alpha^j E_j(u).$$

Then (30) can be reformulated as

$$\mathcal{E}_j(u) = \mathcal{E}_{j+1}(\tilde{u}).$$

The function $\tilde{u}$ is called a harmonic extension of $u$. 

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**Definition 8.** A continuous function $h : V_\ast \to \mathbb{R}$ is called harmonic if it minimizes $\mathcal{E}_j$ at all levels for given boundary values on $V_0$:

$$\mathcal{E}_j(h) = \min \{ \mathcal{E}_j(u) : u|_{V_0} = \rho \}.$$  

According to the Theorem 3.2.4 in [8] for any harmonic function $u$ there exists a unique extension $\tilde{u} \in C(\Gamma)$ such that

$$\tilde{u}|_{V_\ast} = u|_{V_\ast}.$$  

Thus, we identify $u$ with its extension $\tilde{u}$ and think of a harmonic function as a continuous function on $\Gamma$. The maximum and the minimum of the harmonic function are attained at the boundary $V_0$. This assertion is known as the maximum principle [8].

**Definition 9.** A continuous function $\psi : V_\ast \to \mathbb{R}$ is called piecewise harmonic of level $j$ if $\psi \circ F_w$ is harmonic for all $|w| = j$.  

We denote the set of piecewise harmonic functions of level $j$ by $H_j$. These functions minimize $\mathcal{E}_m$ at all levels $m \geq j$ for given boundary values on $V_j$.  

For $f : V_\ast \to \mathbb{R}$ define

$$\mathcal{E}(f) = \lim_{j \to \infty} \mathcal{E}_j(f),$$  

$$\hat{\mathcal{D}} = \{ f : V_\ast \to \mathbb{R}, \mathcal{E}(f) < \infty \}.$$  

If $f \in \hat{\mathcal{D}}$, then it is uniformly continuous on $V_\ast$, hence it has a unique continuous extension to $\Gamma$. Let

$$\mathcal{D} = \{ f \in C(\Gamma) : \mathcal{E}(f) < \infty \}.$$  

Then $(\mathcal{E}, \mathcal{D})$ is regular Dirichlet form on $L_2(\Gamma, \mu)$.  

By effective resistance metric on the set $\Gamma$ we mean a function $R : \Gamma \times \Gamma \to [0, \infty]$ defined by $R(x, x) = 0$ for $x \in \Gamma$ and

$$R(x, y)^{-1} = \inf \{ \mathcal{E}(f) : f(x) = 0, f(y) = 1 \}.$$  

Let $\psi^j_\xi, \xi \in V_j$, be a piecewise harmonic function of level $j$ which equals 1 at $\xi$ and 0 at any other vertex of $V_j$:

$$\psi^j_\xi(x) = \delta_{\xi x} = \begin{cases} 1, & x = \xi \\ 0, & x \in V_j \setminus \{ \xi \} \end{cases}.$$
Note that \( \text{supp } \psi_j^\xi \subset B(\xi, \rho^j) \).

In the case of the unit interval \( I \) piecewise harmonic functions are just piecewise linear functions. In fact, for \( x = \frac{m}{2^j} + \frac{1}{2^j} \in V_j \setminus V_{j-1} \)

\[
\psi_j^x(t) = \begin{cases} 
2^j(t - \frac{m}{2^j}), & \frac{m}{2^j} \leq t < \frac{m+1}{2^j}, \\
2^j(\frac{m+1}{2^j} - t), & \frac{m+1}{2^j} \leq t < \frac{m+2}{2^j}, \\
0, & \text{otherwise},
\end{cases}
\]

and it holds

\[
\left| \psi_j^x(t) - \psi_j^x(s) \right| \leq c |t - s| \quad \text{for all } t, s \in I. \tag{31}
\]

For the Koch curve \( \Gamma \) piecewise harmonic functions \( \tilde{\psi}_j^\xi \) with \( \xi = H(x) \) are the composition \( \psi_j^x \) with the transform \( H^{-1} \) from (27),

\[
\tilde{\psi}_j^x = \psi_j^x \circ H^{-1}.
\]

Taking into account (31) and (28) we get

\[
\left| \tilde{\psi}_j^\xi(\gamma) - \tilde{\psi}_j^\xi(\delta) \right| \leq c |\gamma - \delta|^d, \tag{32}
\]

where \( d = \frac{\ln 4}{\ln 3} \) is the Hausdorff dimension of \( \Gamma \).

In general it was shown in [9] that harmonic functions on \( \Gamma \) are uniformly Lipschitz continuous with respect to the resistance metric \( R(x, y) \). From [3] follows that for a certain class of nested fractals there exist constants \( c, c' > 0 \) such that for all \( x, y \in \Gamma \)

\[
c' |x - y| \frac{\log \frac{4}{3}}{\log \rho} \leq R(x, y) \leq c |x - y| \frac{\log \frac{4}{3}}{\log \rho},
\]

note that \( \frac{\log \frac{4}{3}}{\log \rho} = d_w - d \). Thus piecewise harmonic functions on certain nested fractals satisfy

\[
\left| \psi_j^\xi(x) - \psi_j^\xi(y) \right| \leq c |x - y|^\sigma, \tag{33}
\]

with \( \sigma = d_w - d \). In particular, piecewise harmonic functions on the Sierpinski gasket satisfy

\[
\left| \psi_j^\xi(x) - \psi_j^\xi(y) \right| \leq c |x - y|^{\beta}, \quad \text{for all } x, y \in \text{SG},
\]

where \( \beta = \frac{\ln(5/3)}{\ln 2} \).

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5 Characterization of Besov spaces $B^{s}_{pq}(\Gamma, \mu)$ by piecewise harmonic functions

Let $f \in C(\Gamma)$ and let $P_{n}f$, $n \geq 0$, be the unique piecewise harmonic function in $H_{n}$ which interpolates $f$ at all points in $V_{n}$:

$$P_{0}f = \sum_{\xi \in V_{0}} f(x)\psi_{\xi}^{0},$$

$$P_{n}f = \sum_{\xi \in V_{0}} f(x)\psi_{\xi}^{0} + \sum_{j=1}^{n} \sum_{\xi \in V_{j} \setminus V_{j-1}} c_{\xi}(f)\psi_{\xi}^{j}, \ n \geq 1,$$

with

$$c_{\xi}(f) = f(\xi) - P_{j-1}f(\xi), \quad \xi \in V_{j} \setminus V_{j-1}, \ 1 \leq j \leq n, \quad (34)$$

[8, Definition 3.2.18]. For $\xi \in V_{j} \setminus V_{j-1}$ there is an $\omega \in \Sigma$ such that

$$\xi = \pi(\omega). \quad (35)$$

We define $\Delta(\xi)$ by

$$\Delta(\xi) = \{ \eta \in V_{j-1} : \eta \in F_{\omega_{1}\omega_{2}...\omega_{j-1}}(\Gamma) \},$$

where $\omega$ is chosen according to (35). $\Delta(\xi)$ consists of vertices of $(j - 1)$-simplex that $\xi$ belongs to. It is the same as the one defined in [5, Section 4.1]. Then (34) can be equivalently calculated as

$$c_{\xi}(f) = f(\xi) - \sum_{\eta \in \Delta(\xi)} \alpha_{\xi\eta}f(\eta), \quad \xi \in V_{j} \setminus V_{j-1}, \ 1 \leq j \leq n, \quad (36)$$

with

$$\sum_{\eta \in \Delta(\xi)} \alpha_{\xi\eta} = 1. \quad (37)$$

Let $V_{-1} = \emptyset$ and $P_{-1}f \equiv 0$, then

$$P_{n}f = \sum_{j=0}^{n} \sum_{\xi \in V_{j} \setminus V_{j-1}} c_{\xi}(f)\psi_{\xi}^{j}.$$
From the maximum principle for harmonic functions and the Proposition 1.3.2 in [14] follows that \( P_n f \) tends to \( f \) uniformly on \( \Gamma \) as \( n \to \infty \) and \( f \in C(\Gamma) \) has the unique representation

\[
f = \sum_{j=0}^{\infty} \sum_{\xi \in V_j \setminus V_{j-1}} c_{\xi}(f) \psi^j_{\xi}.
\]

The question arises whether Besov spaces with a certain range of parameters can be characterized by coefficients \( c_{\xi}(f) \). We give an affirmative answer in the following theorem.

\textbf{Theorem 5.} Let \( \Gamma \) be the above \( d \)-set with \( \rho \) as in (26) and \( \sigma \) as in (33). Let

\[
1 < p < \infty \quad \text{and} \quad d \frac{1}{p} < s < \min\{1, \sigma\}.
\]

(38)

Then \( f \in C(\Gamma) \) belongs to \( B^s_p(\Gamma) \) if and only if it can be represented as

\[
f = \sum_{j=0}^{\infty} \sum_{\xi \in V_j \setminus V_{j-1}} c_{\xi}(f) \psi^j_{\xi},
\]

(39)

where

\[
C^s_p(f) = \left( \sum_{j=0}^{\infty} \rho^j \left( \frac{1}{p} \sum_{\xi \in V_j \setminus V_{j-1}} |c_{\xi}(f)|^p \right)^{\frac{1}{p}} \right)^{-\frac{1}{p}} < \infty,
\]

unconditional convergence being in \( C(\Gamma) \). Furthermore,

\[
\|f\|_{B^s_p(\Gamma)} \sim C^s_p(f).
\]

\textbf{Proof.} The idea of the proof is the same as in [5, Theorem 5.1]. Let

\[
a_{j,\xi}(x) = \rho^j (s-d) \psi^j_{\xi}(x), \quad j \in \mathbb{N}_0, \ \xi \in V_j \setminus V_{j-1}.
\]

Then \( a_{j,\xi} \) satisfy (20)-(22). Taking into account that \( C(\Gamma) \subset L_1(\Gamma) \) we get that (39) is an atomic representation of \( f \) and from the Theorem 3 it follows that

\[
\|f\|_{B^s_p(\Gamma)} \leq c C^s_p(f).
\]
To prove the converse, let \( f \in B^s_p(\Gamma) \) and let

\[
f = \sum_{j=0}^{\infty} \sum_{m=1}^{M_j} \lambda_m^j a_{jm}\]

be an atomic decomposition of \( f \) into \((s,p,\sigma)\)-atoms with \( r = \rho \) in (14), (21) and (22) such that

\[
\|\lambda\| \leq c \|f|B^s_p(\Gamma)\|. \tag{40}
\]

Then taking into account (21) and that \( s > \frac{d}{p} \) we get

\[
\left| \sum_{m=1}^{M_j} \lambda_m^j a_{jm} \right| \leq \sup_m |\lambda_m^j| \sum_{m=1}^{M_j} \rho^{i(s-\frac{d}{p})} \leq c \rho^{(s-\frac{d}{p})} \sup_m |\lambda_m^j| \\
\leq c \rho^{(s-\frac{d}{p})} \left( \sum_{m=1}^{M_j} |\lambda_m^j|^p \right)^{\frac{1}{p}}.
\]

The Weierstrass test together with the estimate (40) imply that the series

\[
\sum_{j=0}^{\infty} \sum_{m=1}^{M_j} \lambda_m^j a_{jm}
\]

converges uniformly and it follows

\[
c_\xi(f) = \sum_{j=0}^{\infty} \sum_{m=1}^{M_j} \lambda_m^j c_\xi(a_{jm}).
\]

From the formula (36) together with (37) and the property (21) of \((s,p,\sigma)\)-atoms follows

\[
|c_\xi(a_{jm})| \leq 2 \rho^{i(s-\frac{d}{p})} \tag{41}
\]

Moreover, for \( i > 0 \) the property (22) implies

\[
|c_\xi(a_{jm})| = \left| a_{jm}(\xi) - \sum_{\eta \in \Delta(\xi)} \alpha_{\xi\eta} a_{jm}(\eta) \right| = \left| \sum_{\eta \in \Delta(\xi)} \alpha_{\xi\eta} (a_{jm}(\xi) - a_{jm}(\eta)) \right| \\
\leq \rho^{i\sigma} \rho^{i(s-\sigma-\frac{d}{p})}, \quad \xi \in V_i \setminus V_{i-1}. \tag{42}
\]
Let us split \( c_\xi(f) \) into two parts

\[
c_\xi(f) = \sum_{j=0}^{i} \sum_{m=1}^{M_j} \lambda_m^j c_\xi(a_{jm}) + \sum_{j=i+1}^{\infty} \sum_{m=1}^{M_j} \lambda_m^j c_\xi(a_{jm}) = x_\xi(f) + y_\xi(f).
\]

Taking into account the support condition for atoms (20), we get that for all \( \xi \) and \( j \) the number of atoms such that \( c_\xi(a_{jm}) \neq 0 \) is finite.

First we deal with

\[
X_{i,p} = \left( \sum_{\xi \in V_i \setminus V_{i-1}} |x_\xi(f)|^p \right)^{1/p}.
\]

Note that

\[
\{ \xi \in V_i \setminus V_{i-1} : c_\xi(a_{jm}) \neq 0 \} \subset \{ \xi \in V_i \cap B^\Gamma(\gamma_{jm}, \rho^j) \}.
\]

The balls \( B^\Gamma(\xi, \frac{\rho}{2}) \) corresponding to different \( \xi \in V_i \cap B^\Gamma(\gamma_{jm}, \rho^j) \) are disjoint and for \( j < i \) they are contained in \( B^\Gamma(\gamma_{jm}, 2\rho^j) \). Thus

\[
\sum_{\xi \in V_i \cap B^\Gamma(\gamma_{jm}, \rho^j)} \mu \left( B^\Gamma \left( \xi, \frac{\rho}{2} \right) \right) \leq \mu \left( B^\Gamma \left( \gamma_{jm}, 2\rho^j \right) \right).
\]

Since \( \mu \) is a \( d \)-measure this implies that \( \{ \xi \in V_i \cap B^\Gamma(\gamma_{jm}, \rho^j) \} \) can have at most \( c \left( \frac{\rho}{\rho^j} \right)^d \) elements. Hence

\[
\# \{ \xi \in V_i \setminus V_{i-1} : c_\xi(a_{jm}) \neq 0 \} \leq c\rho^{(j-i)d}, \quad j < i.
\]

By Minkowski’s and Hölder’s inequalities together with (42) follows

\[
X_{i,p} = \left( \sum_{\xi \in V_i \setminus V_{i-1}} |x_\xi(f)|^p \right)^{1/p} \leq \sum_{j=0}^{i} \left( \sum_{\xi \in V_i \setminus V_{i-1}} \left( \sum_{m=1}^{M_j} |\lambda_m^j|^p |c_\xi(a_{jm})|^p \right)^{1/p} \right)^{1/p}
\]

\[
\leq \sum_{j=0}^{i} \sum_{m=1}^{M_j} \left( \sum_{\xi \in V_i \setminus V_{i-1}} |\lambda_m^j|^p |c_\xi(a_{jm})|^p \right)^{1/p} \leq \sum_{j=0}^{i} \sum_{m=1}^{M_j} |\lambda_m^j|^p \rho^{(s-\sigma)^d (j-i)^\frac{d}{p}} \rho^{(j-i)^\frac{d}{p}}
\]

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\[
\leq c \rho^{i(\sigma - \frac{d}{p})} \sum_{j=0}^{i} \rho^{j(s-\sigma)} \left( \sum_{m=1}^{M_j} |\lambda^j_m|^p \right)^{1/p}
\]

and it follows

\[
X_{i,p,s} = \rho^{i(\sigma - \frac{d}{p})} X_{i,p} \leq c \rho^{i(\sigma - s)} \sum_{j=0}^{i} \rho^{j(s-\sigma)} \left( \sum_{m=1}^{M_j} |\lambda^j_m|^p \right)^{1/p}.
\]

Jensen’s inequality implies

\[
X^p_{i,p,s} \leq c \rho^{i(\sigma - s)p} \rho^{j(s-\sigma)(p-1)} \sum_{j=0}^{i} \rho^{j(s-\sigma)} \sum_{m=1}^{M_j} |\lambda^j_m|^p
\]

\[
= c \rho^{j(s-\sigma)} \sum_{j=0}^{i} \rho^{j(s-\sigma)} \sum_{m=1}^{M_j} |\lambda^j_m|^p.
\]

Then

\[
\left( \sum_{i=0}^{\infty} X^p_{i,p,s} \right)^{1/p} \leq c \left( \sum_{i=0}^{\infty} \rho^{i(\sigma - s)} \sum_{j=0}^{i} \rho^{j(s-\sigma)} \sum_{m=1}^{M_j} |\lambda^j_m|^p \right)^{1/p}
\]

\[
= c \left( \sum_{j=0}^{\infty} \left( \sum_{i=j}^{\infty} \rho^{i(s-\sigma)} \sum_{m=1}^{M_j} |\lambda^j_m|^p \right)^{1/p} \sum_{j=0}^{\infty} \sum_{m=1}^{M_j} |\lambda^j_m|^p \right)^{1/p}
\]

\[
= c \| \lambda \|.
\]

To estimate

\[
Y_{i,p} = \left( \sum_{\xi \in V \setminus V_{i-1}} |y_{\xi(f)}|^p \right)^{1/p}
\]

we use Minkowski’s and Hölder’s inequalities together with the property (41). Then we get

\[
Y_{i,p} \leq \sum_{j=i}^{\infty} \left( \sum_{\xi \in V \setminus V_{i-1}} \left( \sum_{m=1}^{M_j} |\lambda^j_m| \sum_{m=1}^{M_j} |c_\xi(a_{jm})|^p \right)^{1/p} \right)
\]

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\[
\begin{align*}
&\leq \sum_{j=1}^{\infty} \sum_{m=1}^{M_j} \left( \sum_{\xi \in V_i \setminus V_{i-1}} |\lambda_m^j|^p |c_{\xi}(a_{jm})|^p \right)^{1/p} \leq c \sum_{j=1}^{\infty} \sum_{m=1}^{M_j} |\lambda_m^j|^{(s-\frac{d}{p})} \\
&\leq c \sum_{j=1}^{\infty} \rho^{(s-\frac{d}{p})} \left( \sum_{m=1}^{M_j} |\lambda_m^j|^p \right)^{1/p}.
\end{align*}
\]

Hence we have
\[
Y_{i,p,s} = \rho^{i(\frac{d}{p}-s)} Y_{i,p} \leq c \rho^{i(\frac{d}{p}-s)} \sum_{j=1}^{\infty} \rho^{j(s-\frac{d}{p})} \left( \sum_{m=1}^{M_j} |\lambda_m^j|^p \right)^{1/p}.
\]

Applying Jensen’s inequality we get
\[
Y_{i,p,s}^p \leq c \rho^{i(\frac{d}{p}-s)} \sum_{j=1}^{\infty} \rho^{j(s-\frac{d}{p})} \left( M_j \sum_{m=1}^{M_j} |\lambda_m^j|^p \right)^{1/p} \leq c \rho^{i(\frac{d}{p}-s)} \sum_{j=1}^{\infty} \rho^{j(s-\frac{d}{p})} \left( \sum_{m=1}^{M_j} |\lambda_m^j|^p \right)^{1/p}.
\]

Then
\[
\begin{align*}
\left( \sum_{i=0}^{\infty} Y_{i,p,s}^p \right)^{1/p} &\leq c \left( \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \rho^{j(\frac{d}{p}-s)} \rho^{j(s-\frac{d}{p})} \left( \sum_{m=1}^{M_j} |\lambda_m^j|^p \right)^{1/p} \right) \\
&\leq c \left( \sum_{j=0}^{\infty} \left( \sum_{i=0}^{j} \rho^{i(\frac{d}{p}-s)} \rho^{i(s-\frac{d}{p})} \sum_{m=1}^{M_j} |\lambda_m^j|^p \right)^{1/p} \right) \leq c \left( \sum_{j=0}^{\infty} \sum_{m=1}^{M_j} |\lambda_m^j|^p \right)^{1/p} = c \|\lambda\|.
\end{align*}
\]

Thus
\[
C_p^s(f) = \left( \sum_{i=0}^{\infty} \rho^{i(\frac{d}{p}-s)} \sum_{\xi \in V_i \setminus V_{i-1}} |c_{\xi}(f)|^p \right)^{1/p} \leq \left( \sum_{i=0}^{\infty} X_{i,p,s}^p \right)^{1/p} + \left( \sum_{i=0}^{\infty} Y_{i,p,s}^p \right)^{1/p} \leq c \|\lambda\| \leq c \|f|B_p^s(\Gamma)\|.
\]
Corollary 1. Let
\[ 1 < p < \infty \text{ and } \frac{d}{p} < s < \min \{1, \sigma\}. \]
The system of functions \(\{\psi^j_\xi, j \in \mathbb{N}_0, \xi \in V_j \setminus V_{j-1}\}\) is an unconditional basis in \(B^s_p(\Gamma)\).

Proof. Let \(f \in B^s_p(\Gamma)\). Then \(f\) has the unique representation
\[ f = \sum_{j=0}^{\infty} \sum_{\xi \in V_j \setminus V_{j-1}} c_\xi(f) \psi^j_\xi \tag{43} \]
with the convergence first being in \(C(\Gamma)\). It is left to show that (43) converges in \(B^s_p(\Gamma)\).

Let us show that the sequence of partial sums
\[ S_n = \sum_{j=0}^{n} \sum_{\xi \in V_j \setminus V_{j-1}} c_\xi(f) \psi^j_\xi \]
is a Cauchy sequence in \(B^s_p(\Gamma)\). For \(n > m\)
\[ \|S_n - S_m|B^s_p(\Gamma)\| = \| \sum_{j=m+1}^{n} \sum_{\xi \in V_j \setminus V_{j-1}} c_\xi(f) \psi^j_\xi |B^s_p(\Gamma)\| \sim \left( \sum_{j=m+1}^{n} \rho^j \sum_{\xi \in V_j \setminus V_{j-1}} |c_\xi(f)|^p \right)^{\frac{1}{p}} \rightarrow 0, \ n, m \rightarrow \infty. \]
Since \(B^s_p(\Gamma)\) is complete, the series (43) converges to \(f\) in \(B^s_p(\Gamma)\). \(\square\)

Corollary 2. Let
\[ \frac{d}{p} < s < \min \{1, \sigma\}, \ 1 < p < \infty \ 1 \leq q < \infty. \]
Then the Theorem 5 remains valid for \(B^s_{pq}(\Gamma, \mu)\).

The proof follows by the same arguments that were used in [7, Section 3.3].
References


