Tempered Radon measures

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Abstract

A tempered Radon measure is a σ-finite Radon measure in \( \mathbb{R}^n \) which generates a tempered distribution. We prove the following assertions. A Radon measure \( \mu \) is tempered if, and only if, there is a real number \( \beta \) such that \( (1 + |x|^2)^\beta \mu \) is finite. A Radon measure is finite if, and only if, it belongs to the positive cone \( \mathcal{B}_0^1(\mathbb{R}^n) \) of \( \mathcal{B}_1^\infty(\mathbb{R}^n) \). Then \( \mu(\mathbb{R}^n) \sim ||\mu|\mathcal{B}_0^1(\mathbb{R}^n)|| \) (equivalent norms).

Key words: Radon measure, tempered distributions, Besov spaces

2000 Mathematics Subject Classification: 42B35, 28C05.

Introduction

A substantial part of fractal geometry and fractal analysis deals with Radon measures in \( \mathbb{R}^n \) (also called fractal measures) with compact support. One may consult [5] and the references given there. In the present paper we clarify the relation between arbitrary σ-finite Radon measure in \( \mathbb{R}^n \), tempered distributions and weighted Besov spaces. It comes out that a σ-finite Radon measure \( \mu \) in \( \mathbb{R}^n \) can be identified with a tempered distribution \( \mu \in \mathcal{S}'(\mathbb{R}^n) \) if and only if there is a real number \( \beta \) such that

\[ \mu_\beta(\mathbb{R}^n) < \infty, \quad \text{where} \quad \mu_\beta = (1 + |x|^2)^\beta \mu. \]
Radon measures $\mu$ with $\mu(\mathbb{R}^n) < \infty$ are called finite. These finite Radon measures can be identified with the positive cone $B^0_{1\infty}(\mathbb{R}^n)$ of the distinguished Besov space $B^0_{1\infty}(\mathbb{R}^n)$ and

$$|||\mu|||_{B^0_{1\infty}(\mathbb{R}^n)} \sim \mu(\mathbb{R}^n)$$

(equivalent norms).

This paper is organised as follows. In Section 1 we collect the definitions and preliminaries. We introduce the well-known weighted Besov spaces $B^s_{pq}(\mathbb{R}^n, \langle x \rangle^\alpha)$ and prove that for fixed $p, q$ with $0 < p, q \leq \infty$

$$S(\mathbb{R}^n) = \bigcap_{\alpha, s \in \mathbb{R}} B^s_{pq}(\mathbb{R}^n, \langle x \rangle^\alpha)$$

and

$$S'(\mathbb{R}^n) = \bigcup_{\alpha, s \in \mathbb{R}} B^s_{pq}(\mathbb{R}^n, \langle x \rangle^\alpha).$$

Although known to specialists we could not find an explicit reference. In Section 2 we prove in the Theorems 1 and 2 the above indicated main results.

1 Definitions and preliminaries

Let $\mathbb{N}$ be the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{R}^n$ be Euclidean $n$-space, where $n \in \mathbb{N}$. Put $\mathbb{R} = \mathbb{R}^1$, whereas $\mathbb{C}$ is the complex plane. Let $S(\mathbb{R}^n)$ be the Schwartz space of all complex-valued, rapidly decreasing, infinitely differentiable functions on $\mathbb{R}^n$. By $S'(\mathbb{R}^n)$ we denote its topological dual, the space of all tempered distributions on $\mathbb{R}^n$. $L^p(\mathbb{R}^n)$ with $0 < p \leq \infty$, is the standard quasi-Banach space with respect to Lebesgue measure, quasi-normed by

$$|||f|||_{L^p(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{\frac{1}{p}}, \quad 0 < p < \infty$$

with the standard modification if $p = \infty$.

If $\varphi \in S(\mathbb{R}^n)$ then

$$\hat{\varphi}(\xi) = F\varphi(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \varphi(x)e^{-ix\xi} \, dx, \quad \xi \in \mathbb{R}^n,$$
denotes the Fourier transform of \( \varphi \). The inverse Fourier transform is given by
\[
\varphi^\vee(x) = F^{-1} \varphi(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \varphi(\xi) e^{ix\xi} \, d\xi, \quad x \in \mathbb{R}^n.
\]
One extends \( F \) and \( F^{-1} \) in the usual way from \( S \) to \( S' \). For \( f \in S'(\mathbb{R}^n) \),
\[
Ff(\varphi) = f(F\varphi), \quad \varphi \in S(\mathbb{R}^n).
\]
Let \( \varphi_0 \in S(\mathbb{R}^n) \) with
\[
\varphi_0(x) = 1, \quad |x| \leq 1 \text{ and } \varphi_0(x) = 0, \quad |x| \geq \frac{3}{2}, \quad (1)
\]
and let
\[
\varphi_k(x) = \varphi_0(2^{-k}x) - \varphi_0(2^{-k+1}x), \quad x \in \mathbb{R}^n, \quad k \in \mathbb{N}. \quad (2)
\]
Then, since
\[
1 = \sum_{j=0}^{\infty} \varphi_j(x) \quad \text{for all } x \in \mathbb{R}^n, \quad (3)
\]
the \( \varphi_j \) form a dyadic resolution of unity in \( \mathbb{R}^n \). \((\varphi_k \hat{f})^\vee\) is an entire analytic function on \( \mathbb{R}^n \) for any \( f \in S'(\mathbb{R}^n) \). In particular, \((\varphi_k \hat{f})^\vee(x)\) makes sense pointwise.

**Definition 1.** Let \( \varphi = \{\varphi_j\}_{j=0}^\infty \) be the dyadic resolution of unity according to \((1)-(3)\), \( s, t \in \mathbb{R} \), \( 0 < p \leq \infty \), \( 0 < q \leq \infty \) and
\[
||f||_{B^s_{pt}(\mathbb{R}^n)}^{\varphi} = \left( \sum_{j=0}^{\infty} 2^{jsq} \left| \left( \varphi_k \hat{f} \right)^\vee \right| L_p(\mathbb{R}^n) \right)^{\frac{1}{q}}
\]
(with the usual modification if \( q = \infty \)). Then the Besov space \( B^s_{pt}(\mathbb{R}^n) \) consists of all \( f \in S'(\mathbb{R}^n) \) such that \( ||f||_{B^s_{pt}(\mathbb{R}^n)}^{\varphi} < \infty \).

We denote by \( L_p(\mathbb{R}^n, \langle x \rangle^\alpha) \), where
\[
\langle x \rangle^\alpha = (1 + |x|^2)^{\alpha/2},
\]
the weighted \( L_p \)-space quasi-normed by
\[
||f||_{L_p(\mathbb{R}^n, \langle x \rangle^\alpha)} = ||\langle \cdot \rangle^\alpha f||_{L_p(\mathbb{R}^n)}. \]
Definition 2. Let \( \varphi = \{ \varphi_j \}_{j=0}^{\infty} \) be the dyadic resolution of unity according to (1)-(3), \( s, \in \mathbb{R}, 0 < p \leq \infty, 0 < q \leq \infty \). Then the weighted Besov space \( B^s_{pq}(\mathbb{R}^n, \langle x \rangle^\alpha) \) is a collection of all \( f \in S'(\mathbb{R}^n) \) such that

\[
\| f \|_{B^s_{pq}(\mathbb{R}^n, \langle x \rangle^\alpha)} = \left( \sum_{j=0}^{\infty} 2^{jsq} \left\| \left( \varphi_k \hat{f} \right)^\vee |L_p(\mathbb{R}^n, \langle x \rangle^\alpha) \right\|^q \right)^{\frac{1}{q}}
\]

(with the usual modification if \( q = \infty \)) is finite.

Remark 1. If \( \alpha = 0 \) then we have the space \( B^s_{pq}(\mathbb{R}^n) \) as introduced in Definition 1. It is also known from [1, Ch. 4.2.2] that the operator \( f \mapsto \langle x \rangle^\alpha f \) is an isomorphic mapping from \( B^s_{pq}(\mathbb{R}^n, \langle x \rangle^\alpha) \) onto \( B^s_{pq}(\mathbb{R}^n) \). In particular,

\[
\| \langle \cdot \rangle^\alpha f \|_{B^s_{pq}(\mathbb{R}^n)} \sim \| f \|_{B^s_{pq}(\mathbb{R}^n, \langle x \rangle^\alpha)}.
\]

Next we review some special properties of weighted Besov spaces.

Proposition 1. For fixed \( 0 < p, q \leq \infty \)

\[
S(\mathbb{R}^n) = \bigcap_{\alpha, s \in \mathbb{R}} B^s_{pq}(\mathbb{R}^n, \langle x \rangle^\alpha)
\]

and

\[
S'(\mathbb{R}^n) = \bigcup_{\alpha, s \in \mathbb{R}} B^s_{pq}(\mathbb{R}^n, \langle x \rangle^\alpha).
\]

Proof. Step 1. The inclusion

\[
S(\mathbb{R}^n) \subset \bigcap_{\alpha, s \in \mathbb{R}} B^s_{pq}(\mathbb{R}^n, \langle x \rangle^\alpha)
\]

is clear.

To prove that any \( f \in \bigcap_{\alpha, s \in \mathbb{R}} B^s_{pq}(\mathbb{R}^n, \langle x \rangle^\alpha) \) belongs to \( S(\mathbb{R}^n) \), it is sufficient to show that for any fixed \( N \in \mathbb{N} \) there are \( \alpha(N) \in \mathbb{R} \) and \( s(N) \in \mathbb{R} \) such that

\[
\sup_{|\beta| \leq N} \sup_{x \in \mathbb{R}^n} \langle x \rangle^{2N} |D^\beta f(x)| \leq c \| f \|_{B^s_{pq}(\mathbb{R}^n, \langle x \rangle^\alpha)}.
\]

For any multiindex \( \beta \) there are polynomials \( P^\beta_\gamma \), \( \deg P^\beta_\gamma \leq 2N \) such that

\[
\langle x \rangle^{2N} D^\beta f(x) = \sum_{\gamma \leq \beta} D^\gamma \left[ (P^\beta_\gamma f)(x) \right].
\]
Hence

\[
\sup_{|\beta| \leq N} \sup_{x \in \mathbb{R}^n} \langle x \rangle^{2N} |D^\beta f(x)| = \sup_{|\beta| \leq N} \sup_{x \in \mathbb{R}^n} \sum_{\gamma \leq \beta} D^\gamma \left( (P^\beta f)(x) \right) \leq \\
\sup_{|\beta| \leq N} \sum_{|\gamma| \leq N} \sup_{x \in \mathbb{R}^n} |D^\gamma \left( (P^\beta f)(x) \right) | \leq \sup_{|\beta| \leq N} \sum_{|\gamma| \leq N} \left| P^\beta f \right| \left| C^N(\mathbb{R}^n) \right| .
\]  \tag{5}

Due to the embedding theorems [3, Ch. 2.7.1]

\[
\left| P^\beta f \right| \left| C^N(\mathbb{R}^n) \right| \leq c \left| \frac{P^\beta}{\langle x \rangle^{2N}} \left( \langle x \rangle^{2N} f \right) \right| \left| B_{pq}^{N+\frac{n}{p}+\varepsilon}(\mathbb{R}^n) \right|,
\]  \tag{6}

for any \( \varepsilon > 0 \). \( \frac{P^\beta}{\langle x \rangle^{2N}} \) is a pointwise multiplier for \( B_{pq}^{N+\frac{n}{p}+\varepsilon}(\mathbb{R}^n) \), [3, Ch. 2.8.2], therefore

\[
\left| \frac{P^\beta}{\langle x \rangle^{2N}} \left( \langle x \rangle^{2N} f \right) \right| \left| B_{pq}^{N+\frac{n}{p}+\varepsilon}(\mathbb{R}^n) \right| \leq \\
c \left| \frac{P^\beta}{\langle x \rangle^{2N}} \left( \langle x \rangle^{2N} f \right) \right| \left| \langle x \rangle^{2N} f \right| \left| B_{pq}^{N+\frac{n}{p}+\varepsilon}(\mathbb{R}^n) \right| .
\]  \tag{7}

According to Remark 1

\[
\left| \langle x \rangle^{2N} f \right| \left| B_{pq}^{N+\frac{n}{p}+\varepsilon}(\mathbb{R}^n) \right| \sim \left| f \right| \left| B_{pq}^{N+\frac{n}{p}+\varepsilon}(\mathbb{R}^n, \langle x \rangle^{2N}) \right| .
\]  \tag{8}

Combining (5), (6), (7), (8), one gets

\[
\sup_{|\beta| \leq N} \sup_{x \in \mathbb{R}^n} \langle x \rangle^{2N} |D^\beta f(x)| \leq c \sum_{|\gamma| \leq N} \left| \langle x \rangle^{2N} f \right| \left| B_{pq}^{N+\frac{n}{p}+\varepsilon}(\mathbb{R}^n, \langle x \rangle^{2N}) \right| \leq \\
c \left| f \right| \left| B_{pq}^{N+\frac{n}{p}+\varepsilon}(\mathbb{R}^n, \langle x \rangle^{2N}) \right| .
\]  \tag{9}

and it follows (4).

Step 2. Let \( 1 < p \leq \infty, 1 < q \leq \infty \) and let \( p' \) and \( q' \) be defined in the standard way by

\[
\frac{1}{p} + \frac{1}{p'} = 1, \quad \frac{1}{q} + \frac{1}{q'} = 1.
\]

The inclusion

\[
\bigcup_{\alpha, s \in \mathbb{R}} B_{pq}^{s}(\mathbb{R}^n, \langle x \rangle^\alpha) \subseteq S'(\mathbb{R}^n)
\]

5
is evident.
As far as the opposite inclusion is concerned, we recall that \( f \in S'(\mathbb{R}^n) \)
if and only if there are \( l \in \mathbb{N} \) and \( m \in \mathbb{N} \) such that
\[
|f(\varphi)| \leq c \sup_{|\alpha| \leq m} \sup_{x \in \mathbb{R}^n} \langle x \rangle^l |D^\alpha \varphi(x)|,
\]
for all \( \varphi \in S(\mathbb{R}^n) \). By (9)
\[
\sup_{|\alpha| \leq m} \sup_{x \in \mathbb{R}^n} \langle x \rangle^l |D^\alpha \varphi(x)| \leq c \left| \varphi |B_{p'q'}^{m+n} (\mathbb{R}^n, \langle x \rangle^l) \right|.
\]
According to our choice of \( p \) and \( q \), it follows that \( 1 \leq p' < \infty \) and \( 1 \leq q' < \infty \). Thus by [3, Ch. 2.11.2]
\[
f \in \left( B_{p'q'}^{m+n+\varepsilon} (\mathbb{R}^n, \langle x \rangle^l) \right)' = B_{pq}^{-(m+n+\varepsilon)} (\mathbb{R}^n, \langle x \rangle^{-l}) .
\]
This means
\[
S'(\mathbb{R}^n) \subset \bigcup_{\alpha, s \in \mathbb{R}} B_{pq}^s (\mathbb{R}^n, \langle x \rangle^{\alpha}).
\]
Step 3. Let \( 0 < p \leq 1, 1 < q \leq \infty \). By the arguments above, for \( f \in S'(\mathbb{R}^n) \) there are \( \alpha \in \mathbb{R} \) and \( s \in \mathbb{R} \) such that
\[
f \in B_{\infty q}^s (\mathbb{R}^n, \langle x \rangle^{\alpha}).
\]
We want to show that
\[
f \in B_{pq}^s (\mathbb{R}^n, \langle x \rangle^{\alpha-\gamma}), \quad \gamma > \frac{n}{p}.
\]
Indeed,
\[
\|f|B_{pq}^s (\mathbb{R}^n, \langle x \rangle^{\alpha-\gamma})\| = \left( \sum_{j=0}^{\infty} 2^{jsq} \left| \langle x \rangle^{\alpha-\gamma} (\varphi_j \hat{f}) \right|^q L_p(\mathbb{R}^n) \right)^{\frac{1}{q}} \leq \left( \sum_{j=0}^{\infty} 2^{jsq} \sup_{x \in \mathbb{R}^n} \left| \langle x \rangle^{\alpha} (\varphi_j \hat{f}) \right|^q \left( \int_{\mathbb{R}^n} (\langle x \rangle^{-\gamma p} dx) \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \leq c \|f|B_{\infty q}^s (\mathbb{R}^n, \langle x \rangle^{\alpha})\| .
\]
Step 4. When \(0 < q \leq 1\), first we may find \(\alpha \in \mathbb{R}\) and \(s \in \mathbb{R}\) such that
\[
f \in B_{pq}^s(\mathbb{R}^n, \langle x \rangle^\alpha),
\]
\(q^* > 1\), and then use the fact that
\[
B_{pq}^s(\mathbb{R}^n, \langle x \rangle^\alpha) \subset B_{pq}^{s-\varepsilon}(\mathbb{R}^n, \langle x \rangle^\alpha), \quad \varepsilon > 0.
\]

Next we recall some notation. A measure \(\mu\) is called \(\sigma\)-finite in \(\mathbb{R}^n\) if for any \(R > 0\),
\[
\mu(\{x : |x| < R\}) < \infty.
\]
A measure \(\mu\) is a Radon measure if all Borel sets are \(\mu\) measurable and
(i) \(\mu(K) < \infty\) for compact sets \(K \subset \mathbb{R}^n\),
(ii) \(\mu(V) = \sup \{\mu(K) : K \subset V\text{ is compact}\}\) for open sets \(V \subset \mathbb{R}^n\),
(iii) \(\mu(A) = \inf \{\mu(V) : A \subset V, V \text{ is open}\}\) for \(A \subset \mathbb{R}^n\).

Let \(\mu\) be a positive Radon measure in \(\mathbb{R}^n\). Let \(T_\mu\),
\[
T_\mu : \varphi \mapsto \int_{\mathbb{R}^n} \varphi(x) \mu(dx), \quad \varphi \in S(\mathbb{R}^n),
\]
be the linear functional generated by \(\mu\).

**Definition 3.** A positive Radon measure \(\mu\) is said to be tempered if \(T_\mu \in S'(\mathbb{R}^n)\).

**Proposition 2.** Let \(\mu^1\) and \(\mu^2\) be two tempered Radon measures. Then
\[
T_{\mu^1} = T_{\mu^2} \text{ in } S'(\mathbb{R}^n) \text{ if, and only if, } \mu^1 = \mu^2.
\]

**Proof.** The Proposition is valid by the arguments in [5, p. 80].

This justifies the identification of \(\mu\) and correspondent tempered distribution \(T_\mu\) and we may write \(\mu \in S'(\mathbb{R}^n)\).

**Definition 4.** \(f \in S'(\mathbb{R}^n)\) is called a positive distribution if
\[
f(\varphi) \geq 0 \text{ for any } \varphi \in S(\mathbb{R}^n) \text{ with } \varphi \geq 0.
\]

If \(f \in L^1_{\text{loc}}(\mathbb{R}^n)\) then \(f \geq 0\) means \(f(x) \geq 0\) almost everywhere.
Remark 2. If $f$ is a positive distribution, then $f \in C_0(\mathbb{R}^n)'$ and it follows from the Radon–Riesz theorem that there is a tempered Radon measure $\mu$ such that

$$f(\varphi) = \int_{\mathbb{R}^n} \varphi(x) \mu(dx)$$

[2, p. 61/62, 71, 75].

2 Main assertions

Our next result refers to tempered measures.

Theorem 1. (i) A Radon measure $\mu$ in $\mathbb{R}^n$ is tempered if, and only if, there is a real number $\beta$ such that $\langle x \rangle^\beta \mu$ is finite.

(ii) Let $\mu$ be a tempered Radon measure in $\mathbb{R}^n$. Let $j \in \mathbb{N}$,

$$A_j = \{ x : 2^{j-1} \leq |x| \leq 2^{j+1} \}, \quad A_0 = \{ x : |x| \leq 2 \}.$$

Then for some $c > 0$, $\alpha \geq 0$,

$$\mu(A_k) \leq c 2^{k\alpha} \text{ for all } k \in \mathbb{N}_0.$$  

Proof. Step 1. First we prove part (ii). Suppose that the assertion does not hold. Then for $c = 1$ and $l \in \mathbb{N}$ there is $k_l \in \mathbb{N}_0$ such that

$$\mu(A_{k_l}) > 2^{k_l}.$$  

As soon as it is found one $k_l$ with (10), it follows that there are infinitely many $k_l^m$, $m \in \mathbb{N}$ that satisfy (10).

With $j \in \mathbb{N}$,

$$A_j^* = \{ x : 2^{j-2} \leq |x| \leq 2^{j+2} \}, \quad A_0^* = \{ x : |x| \leq 4 \}.$$

For $l = 1$ take any of $k_1^m$, let it be $k_1$. For $l = 2$ choose $k_2 \gg k_1$ in such a way that $A_{k_1}^*$ and $A_{k_2}^*$ have an empty intersection. For arbitrary $l \in \mathbb{N}$ take

$$k_l \gg k_{l-1} \text{ and } A_{k_{l-1}}^* \cap A_{k_l}^* = \emptyset.$$  

Let $\varphi_0$ be a $C^\infty$ function on $\mathbb{R}^n$ with

$$\varphi_0(x) = 1, \quad |x| \leq 2 \text{ and } \varphi_0(x) = 0, \quad |x| \geq 4.$$
Let $k \in \mathbb{N}$ and

$$\varphi_k(x) = \varphi_0(2^{-k}x) - \varphi_0(2^{-k+3}x), \quad x \in \mathbb{R}^n.$$  

Then we have

$$\text{supp } \varphi_k \subset A^*_k$$  

and

$$\varphi_k(x) = 1, \quad x \in A_k.$$  

Let

$$\varphi(x) = \sum_{l=1}^{\infty} 2^{-lk} \varphi_{kl}(x).$$  

For any fixed $N \in \mathbb{N}_0$

$$\sup_{|\alpha| \leq N} \sup_{x \in \mathbb{R}^n} (1 + |x|)^2 \left| D^\alpha \varphi(x) \right| =$$

$$= \sup_{|\alpha| \leq N} \sup_{x \in \mathbb{R}^n} (1 + |x|)^2 \left| D^\alpha \left( \sum_{l=1}^{\infty} 2^{-lk} \varphi_{kl}(x) \right) \right| \leq$$

$$\leq \sup_{l \in \mathbb{N}} \sup_{|\alpha| \leq N} 2^{-lk} 2^{-|\alpha|k} 2^{|\alpha|} (1 + |x|)^2 N \left| (D^\alpha \varphi_1)(2^{-k+1}x) \right|.$$  

The last inequality holds, since the functions $\varphi_{kl}$ have disjoint supports. With the change of variables

$$x' = 2^{-k+1}x$$

one gets

$$\sup_{|\alpha| \leq N} \sup_{x \in \mathbb{R}^n} (1 + |x|)^2 \left| D^\alpha \varphi(x) \right| \leq$$

$$\leq \sup_{l \in \mathbb{N}} \sup_{|\alpha| \leq N} 2^{-lk} 2^{-|\alpha|k} 2^{|\alpha|} 2^{(k-1)N} \sup_{x \in \mathbb{R}^n} (1 + |x|)^2 N \left| (D^\alpha \varphi_1)(x) \right| \leq$$

$$\leq c \sup_{l \in \mathbb{N}} \sup_{|\alpha| \leq N} 2^{-lk} 2^{-(l+|\alpha|)2N} \leq c \sup_{l \in \mathbb{N}} 2^{-k(l-2N)}.$$  

Since $N$ is fixed and $l$ is tending to infinity, $2^{-k(l-2N)}$ is bounded. Thus $\varphi \in S(\mathbb{R}^n)$.

According to the definition of tempered Radon measures

$$\int_{\mathbb{R}^n} \psi(x) \mu(dx) < +\infty$$
for any $\psi \in S(\mathbb{R}^n)$, but
\[
\int_{\mathbb{R}^n} \varphi(x) \mu(dx) \geq \sum_{l=1}^{\infty} \int_{A_{k_l}} \varphi(x) \mu(dx) \geq \sum_{l=1}^{\infty} 2^{-l k_l} 2^{k_l} = +\infty.
\]

This means that our assertion (10) is false.

Step 2. We prove part (i). Since $\langle x \rangle^\beta \mu$ is finite it tempered. Then $\mu$ is also tempered. To prove the other direction we take $\beta = -(\alpha + 1)$. Then we get
\[
\langle \cdot \rangle^\beta \mu(\mathbb{R}^n) = \int_{\mathbb{R}^n} \langle x \rangle^{-(\alpha+1)} \mu(dx) \leq \sum_{k=0}^{\infty} \int_{A_k} \langle x \rangle^{-(\alpha+1)} \mu(dx) \leq c \sum_{k=0}^{\infty} 2^{-k(\alpha+1)} \int_{A_k} \mu(dx) \leq c \sum_{k=0}^{\infty} 2^{-k(\alpha+1)} 2^{k\alpha} < \infty.
\]

In order to characterize finite Radon measures we define the positive cone $B^+_{pq}(\mathbb{R}^n)$ as the collection of all positive $f \in B^+_p(\mathbb{R}^n)$.

**Theorem 2.** Let $M(\mathbb{R}^n)$ be the collection of all finite Radon measures. Then
\[
M(\mathbb{R}^n) = B^+_0(\mathbb{R}^n)
\]
and
\[
\mu(\mathbb{R}^n) \sim \|\mu|B^0_{1,\infty}(\mathbb{R}^n)\|, \quad \mu \in M(\mathbb{R}^n).
\]

**Proof.** By the proof in [5, p.82/83], Proposition 1.127,
\[
\|\mu|B^0_{1,\infty}(\mathbb{R}^n)\| \leq \mu(\mathbb{R}^n) \text{ if } \mu \in M(\mathbb{R}^n).
\]

In order to prove the converse inequality, one use the characterisation of Besov spaces via local means. Let $k_0$ be a $C^\infty$ non-negative function with
\[
supp k_0 \subset \{x : |x| \leq 1\} \text{ and } k_0(0) \neq 0.
\]
If $f \in B^0_{1,\infty}(\mathbb{R}^n)$, then $f = \mu$ is a tempered measure. By the Theorem 1.10 in [5, p. 10]
\[
\|\mu|B^0_{1,\infty}(\mathbb{R}^n)\| \geq c \|k_0(1, \mu)|L^1(\mathbb{R}^n)\| = c \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} k_0(x-y) \, d\mu(y) \, dx.
\]
Applying Fubini’s theorem, one gets
\[ \left| \mu \right|_{B^{0}_1(\mathbb{R}^n)} \geq c \mu(\mathbb{R}^n). \]

\[ \square \]

**Corollary 1.** Let \( f \in L_1(\mathbb{R}^n) \) and \( f(x) \geq 0 \) almost everywhere. Then
\[ ||f||_{L_1(\mathbb{R}^n)} \sim ||f|B^{0}_{1\infty}(\mathbb{R}^n)||. \]

**Proof.** Let \( \mu = f \mu_L \), where \( \mu_L \) is the Lebesgue measure. Then
\[ \mu(\mathbb{R}^n) = \int_{\mathbb{R}^n} f(x) \mu_L(dx) = ||f|L_1(\mathbb{R}^n)|| \]
and
\[ ||\mu|B^{0}_{1\infty}(\mathbb{R}^n)|| = ||f|B^{0}_{1\infty}(\mathbb{R}^n)||. \]

From (11) follows the statement in the Corollary. \( \square \)

The question arises whether Corollary 1 can be extended to all \( f \in L_1(\mathbb{R}^n) \). We have
\[ L_1(\mathbb{R}^n) \hookrightarrow B^{0}_{1\infty}(\mathbb{R}^n), \text{ hence } ||f|B^{0}_{1\infty}(\mathbb{R}^n)|| \leq c ||f|L_1(\mathbb{R}^n)|| \]
for all \( f \in L_1(\mathbb{R}^n) \). But the converse is not true even for functions \( f \in L_1(\mathbb{R}^n) \) with compact support in the unit ball.

**Proposition 3.** There are functions \( f_j \in L_1(\mathbb{R}^n) \) with
\[ \text{supp } f_j \subset \{ y : |y| \leq 1 \}, \quad j \in \mathbb{N}, \]
such that \( \{ f_j \} \) is a bounded set in \( B^{0}_{1\infty}(\mathbb{R}^n) \), but
\[ ||f_j|L_1(\mathbb{R}^n)|| \rightarrow \infty \text{ if } j \rightarrow \infty. \]

**Proof.** We may assume \( n = 1 \).

Let \( a \in C^1(\mathbb{R}) \) be an odd function with
\[ \text{supp } a \subset \{ x : |x| \leq 2 \}, \quad a(x) \geq 0, \quad x \geq 0 \]
and
\[ \max_{-2 \leq x \leq 2} |a(x)| = |a(-1)| = a(1) = 1. \]
If $c = \max_{-2 \leq x \leq 2} |a'(x)|$, then $c \geq 1$. Define $a_0 \in C^1(\mathbb{R})$ by

$$a_0(x) = c^{-1}a(x).$$

Then one has for any $x \in \mathbb{R}$,

$$|a_0(x)| \leq c^{-1} \leq 1, \quad |a_0'(x)| \leq 1 \quad \text{and} \quad \int_{\mathbb{R}} a_0(x) \, dx = 0.$$

Define a function $a_\nu, \nu \in \mathbb{N}$, by

$$a_\nu(x) = 2^\nu a_0(2^\nu x).$$

Then

$$\text{supp } a_\nu \subset [-2^{-\nu+1}, 2^{-\nu+1}]$$

and

$$|a_\nu(x)| \leq c^{-1}2^\nu, \quad |a_\nu'(x)| \leq 2^{2\nu}, \quad \int_{\mathbb{R}} a_\nu(x) \, dx = 0.$$

According to the Definition 1.15 in [5, p. 12], $a_0$ is $1_1$-atom and $a_\nu$ are $(0, 1)_{1,1}$-atoms. It follows from Theorem 13.8 in [4] that $\sum_{\nu=1}^\infty a_\nu(x)$ converges in $S'(\mathbb{R}^n)$ and represents an element of $B_1^0(\mathbb{R}^n)$. Let $f \equiv \sum_{\nu=1}^\infty a_\nu$.

Let

$$f_j(x) = \sum_{\nu=1}^j a_\nu(x).$$

Then $\text{supp } f_j \subset [-1, 1]$,

$$||f_j||_{L_1(\mathbb{R}^n)} \geq \int_0^{+\infty} f_j(x) \, dx = \int_0^{+\infty} \sum_{\nu=1}^j a_\nu(x) \, dx = \int_0^{+\infty} a_0(x) \, dx \to \infty, \quad j \to \infty.$$

On the other hand one has by the above atomic argument

$$||f_j||_{B_1^0(\mathbb{R})} \leq 1 \quad \text{for } j \in \mathbb{N}.$$
**Corollary 2.** Not any characteristic function of a measurable subset of $\mathbb{R}^n$ is a pointwise multiplier in $B_{1,\infty}^0(\mathbb{R}^n)$.

**Proof.** Let $f \in L_1(\mathbb{R}^n)$ real. Let $M_+$ be a set of points $x$ such that $f(x) \geq 0$ and $M_- = \{x : f(x) < 0\}$. Then

$$||f||_{L_1(\mathbb{R}^n)} = ||\chi_{M_+} f||_{L_1(\mathbb{R}^n)} + ||\chi_{M_-} f||_{L_1(\mathbb{R}^n)},$$

where $\chi_{M_+}$, $\chi_{M_-}$ are characteristic functions of sets $M_+$ and $M_-$ respectively. One may apply Corollary 1 to the functions $\chi_{M_+} f$ and $\chi_{M_-} f$ and get

$$||f||_{L_1(\mathbb{R}^n)} \leq c \left( ||\chi_{M_+} f||_{B_{1,\infty}^0(\mathbb{R}^n)} + c ||\chi_{M_-} f||_{B_{1,\infty}^0(\mathbb{R}^n)}\right).$$

If any characteristic function of a set in $\mathbb{R}^n$ would be a pointwise multiplier in $B_{1,\infty}^0(\mathbb{R}^n)$, then

$$||\chi_{M_+} f||_{B_{1,\infty}^0(\mathbb{R}^n)} \leq c ||f||_{B_{1,\infty}^0(\mathbb{R}^n)} \quad \text{and} \quad ||\chi_{M_-} f||_{B_{1,\infty}^0(\mathbb{R}^n)} \leq c ||f||_{B_{1,\infty}^0(\mathbb{R}^n)},$$

hence

$$||f||_{L_1(\mathbb{R}^n)} \leq c ||f||_{B_{1,\infty}^0(\mathbb{R}^n)}.$$

Since for any function $f \in L_1(\mathbb{R}^n)$ holds

$$||f||_{B_{1,\infty}^0(\mathbb{R}^n)} \leq c ||f||_{L_1(\mathbb{R}^n)}||, $$

one gets

$$||f||_{L_1(\mathbb{R}^n)} \sim ||f||_{B_{1,\infty}^0(\mathbb{R}^n)}||, \quad \text{for real } f \in L_1(\mathbb{R}^n).$$

This can be also extended to complex functions $f \in L_1(\mathbb{R}^n)$. But according to the Proposition 3 this is not true. \qed

**References**


