Coorbit Space Theory for Quasi-Banach Spaces

Holger Rauhut
NuHAG, Faculty of Mathematics, University of Vienna
Nordbergstrasse 15, A-1090 Wien, Austria
rauhut@ma.tum.de

July 8, 2005

Abstract

We generalize the classical coorbit space theory developed by Feichtinger and Gröchenig to quasi-Banach spaces. As a main result we provide atomic decompositions for coorbit spaces defined with respect to quasi-Banach spaces. These atomic decompositions are used to prove fast convergence rates of best n-term approximation schemes. We apply the abstract theory to time-frequency analysis of modulation spaces $M_m^{p,q}$, $0 < p, q \leq \infty$.

AMS subject classification: 42C15, 42C40, 46A16, 46E10, 46E30

Key Words: Coorbit spaces, quasi-Banach spaces, Wiener amalgam spaces, modulation spaces, atomic decompositions, Gabor frames, non-linear approximation, Lorentz spaces

1 Introduction

Coorbit space theory was originally developed by Feichtinger and Gröchenig [8, 9, 10, 13] in the late 1980's with the aim to provide a unified and group-theoretical approach to function spaces and their atomic decompositions. In particular, this theory covers the homogeneous Besov and Triebel-Lizorkin spaces and their wavelet-type atomic decompositions, as well as the modulation spaces and their Gabor-type decompositions. Recently, there has been some activity to provide generalizations to other settings than the classical one of integrable group representations [1, 2, 11, 20, 21].

All the approaches done so far cover only the case of Banach spaces. For certain applications such as non-linear approximation, however, it is useful to consider also the case of quasi-Banach spaces. For instance this would allow to describe also modulation spaces $M^{p,q}_m$ with $p < 1$ or $q < 1$, or Hardy spaces H^p with $p < 1$, as coorbit spaces. In [8] it is remarked that such an extension of coorbit space theory to quasi-Banach spaces would be interesting, but it seems that nothing concrete has been done since then.

So this paper deals with such an extension of the classical coorbit space theory. Our starting point is an integrable representation π of some locally compact group $\mathcal{G}$ on some Hilbert space $\mathcal{H}$. Associated to π is the abstract wavelet transform $V_g f(x) = \langle f, \pi(x)g\rangle$. The crucial ingredient in coorbit space theory is the reproducing formula for V_g, see (4.2), which uses the group convolution on $\mathcal{G}$. Thus, it is essential to have a convolution relations for certain quasi-Banach spaces Y on $\mathcal{G}$. Unfortunately, even for the natural choice $Y = L^p(\mathcal{G})$, $0 < p < 1$, no convolution relation is available. In order to overcome this problem we work with Wiener amalgam spaces $W(L^\infty, Y)$ with local component L^∞ instead of Y itself. Convolution relations for such spaces, where Y is allowed to be a quasi-Banach space, were shown recently by the author in [22].

Under some technical assumption on the representation, the coorbit spaces $\mathcal{C}(Y)$ are defined as retract of the Wiener amalgam space $W(L^\infty, Y)$ via the abstract wavelet transform, i.e., $\mathcal{C}(Y) = \{f, V_g f \in W(L^\infty, Y)\}$. We will prove that $\mathcal{C}(Y)$ is indeed a quasi-Banach space that is independent of the choice of g. Moreover, analogously as in [8, 9, 13] we will provide atomic decompositions of $\mathcal{C}(Y)$ of the form $\{\pi(x_i)g\}_{i\in I}$, where $(x_i)_{i\in I}$ is a suitable point set in the group.

Our results are applicable to time-frequency analysis on modulation spaces $M^{p,q}_m$, $0 < p, q \leq \infty$, introduced by Feichtinger [7], see also [23, 12] for the case $p, q < 1$. Hereby, we improve or give alternative proofs to some of the results of Galperin and Samarah in [12]. In particular, we show that regular Gabor frame expansions with a Schwartz class window automatically extend from $L^2(\mathbb{R}^d)$ to all modulation spaces $M^{p,q}_m$, $0 < p, q \leq \infty$ (with moderate weight functions m of polynomial growth).

We remark that the abstract theory applies also to homogeneous (weighted) Besov spaces $\dot{B}^s_{p,q}$ and Triebel-Lizorkin spaces $\dot{F}^s_{p,q}$, $0 < p, q \leq \infty$. We postpone a detailed discussion to a subsequent contribution.

The paper is organized as follows. In Section 2 we introduce some notation and prerequisites. Section 3 recalls recent results from [22] on Wiener amalgam spaces with respect to quasi-Banach spaces including convolution relations. Then in Section 4 we introduce the coorbit spaces and show their basic properties. The atomic decompositions of the coorbit spaces will be provided in Section 5 and Section 6 deals with the question whether the coorbit space admits also a characterization by Y itself rather than by $W(L^\infty, Y)$. Section 7 investigates the approximation rates of the best

n-term approximation with elements of the atomic decomposition. Here, Lorentz spaces play a key role. Finally, we apply our abstract results to time-frequency analysis of modulation spaces in Section 8.

2 Prerequisites

Let $\mathcal{G}$ be a locally compact group with identity e. Integration on $\mathcal{G}$ will always be with respect to the left Haar measure. We denote by $L_xF(y) = F(x^{-1}y)$ and $R_xF(y) = F(yx)$, $x, y \in \mathcal{G}$, the left and right translation operators. Furthermore, let Δ be the Haar-module on $\mathcal{G}$. For a Radon measure μ we introduce the operator $(A_x\mu)(k) = \mu(R_xk)$, $x \in \mathcal{G}$, for a continuous function k with compact support. We may identify a function $F \in L^1$ with a measure $\mu_F \in M$ by $\mu_F(k) = \int F(x)k(x)dx$. Then it clearly holds $A_xF = \Delta(x^{-1})R_{x^{-1}}F$. Further, we define the involutions $F^\vee(x) = F(x^{-1})$, $F^\nabla(x) = \overline{F(x^{-1})}$, $F^*(x) = \Delta(x^{-1})\overline{F(x^{-1})}$.

A quasi-norm $\|\cdot\|$ on some linear space Y is defined in the same way as a norm, with the only difference that the triangle inequality is replaced by $\|f + g\| \leq C(\|f\| + \|g\|)$ with some constant $C \geq 1$. It is well-known, see e.g. [3, p. 20] or [19], that there exists an equivalent quasi-norm $\|\cdot|Y\|$ on Y and an exponent p with $0 < p \leq 1$ such that $\|\cdot|Y\|$ satisfies the p-triangle inequality, i.e., $\|f + g|Y\|^p \leq \|f|Y\|^p + \|g|Y\|^p$. We can choose $p = 1$ if and only if Y is a normed space. We always assume in the sequel that such a p-norm on Y is chosen and denote it by $\|\cdot|Y\|$. If Y is complete with respect to the topology defined by the metric $d(f, g) = \|f - g|Y\|^p$ then it is called a quasi-Banach space.

A quasi-Banach space of measurable functions on $\mathcal{G}$ is called solid if $F \in Y$, G measurable and satisfying $|G(x)| \leq |F(x)|$ a.e. implies $G \in Y$ and $\|G|Y\| \leq \|F|Y\|$. The Lebesgue spaces $L^p(\mathcal{G})$, $0 < p \leq \infty$, provide natural examples of solid quasi-normed spaces on $\mathcal{G}$, and the usual quasi-norm in $L^p(\mathcal{G})$ is a p-norm if $0 < p \leq 1$. If w is some positive measurable weight function on $\mathcal{G}$ then we further define $L^p_w = \{F \text{ measurable }, Fw \in L^p\}$ with $\|F|L^p_w\| := \|Fw|L^p\|$. A continuous weight w is called submultiplicative if $w(xy) \leq w(x)w(y)$ for all $x, y \in \mathcal{G}$. Further, another weight m is called w-moderate if $m(xyz) \leq w(x)m(y)w(z)$, $x, y, z \in \mathcal{G}$. It is easy to see that L^p_m is invariant under left and right translations if m is w-moderate.

For a quasi-Banach space $(B, \|\cdot|B\|)$ we denote the quasi-norm of a bounded operator $T : B \to B$ by $\|\!|T|B\|\!|$. The symbol $A \asymp B$ indicates throughout the paper that there are constants $C_1, C_2 > 0$ such that $C_1A \leq B \leq C_2A$ (independently on other expressions on which A, B might depend). The symbol C will always denote a generic constant whose precise value might differ at different occurences.

3 Wiener Amalgam Spaces

Let B be one of the spaces $L^\infty(\mathcal{G}), L^1(\mathcal{G})$ or $M(\mathcal{G})$, the space of complex Radon measures. Choose some relatively compact neighborhood Q of $e \in \mathcal{G}$. We define the control function by

$$K(F, Q, B)(x) := \|(L_x \chi_Q)F|B\|, \quad x \in \mathcal{G}, \tag{3.1}$$

where F is locally contained in B, in symbols $F \in B_{loc}$. Further, let Y be some solid quasi-Banach space of functions on $\mathcal{G}$ containing the characteristic function of any compact subset of $\mathcal{G}$. The **Wiener amalgam space** $W(B, Y)$ is then defined by

$$W(B, Y) := W(B, Y, Q) := \{F \in B_{loc},\, K(F, Q, B) \in Y\}$$

with quasi-norm

$$\|F|W(B, Y, Q)\| := \|K(F, Q, B)|Y\|. \tag{3.2}$$

This is indeed a p-norm with p being the exponent of the quasi-norm of Y. By $W(C_0, Y)$ we denote the closed subspace of $W(L^\infty, Y)$ consisting of continuous functions.

We also need certain discrete sets in $\mathcal{G}$.

Definition 3.1. *Let $X = (x_i)_{i \in I}$ be some discrete set of points in $\mathcal{G}$ and V, W relatively compact neighborhoods of e in $\mathcal{G}$.*

- *(a) X is called V-dense if $\mathcal{G} = \bigcup_{i \in I} x_i V$.*
- *(b) X is called relatively separated if for all compact sets $K \subset \mathcal{G}$ there exists a constant C_K such that $\sup_{j \in I} \#\{i \in I,\, x_i K \cap x_j K \neq \emptyset\} \leq C_K$.*
- *(c) X is called V-well-spread (or simply well-spread) if it is both relatively separated and V-dense for some V.*

The existence of V-well-spread sets for arbitrarily small V is proven in [6], see also [20, 21] for a generalization. Given a well-spread family $X = (x_i)_{i \in I}$, a relatively compact neighborhood Q of $e \in \mathcal{G}$ and Y, we define the sequence space

$$Y_d := Y_d(X) := Y_d(X, Q) := \{(\lambda_i)_{i \in I}, \sum_{i \in I} |\lambda_i| \chi_{x_i Q} \in Y\}, \tag{3.3}$$

with natural norm $\|(\lambda_i)_{i \in I}|Y_d\| := \|\sum_{i \in I} |\lambda_i| \chi_{x_i Q}|Y\|$. Hereby, $\chi_{x_i Q}$ denotes the characteristic function of the set $x_i Q$. If the quasi-norm of Y is a p-norm, $0 < p \leq 1$, then also Y_d has a p-norm.

The following concept will also be very useful.

Definition 3.2. *Suppose U is a relatively compact neighborhood of $e \in \mathcal{G}$. A collection of functions $\Psi = (\psi_i)_{i\in I}, \psi_i \in C_0(\mathcal{G})$, is called bounded uniform partition of unity of size U (for short U-BUPU) if the following conditions are satisfied:*

(1) $0 \leq \psi_i(x) \leq 1$ *for all* $i \in I$, $x \in \mathcal{G}$,

(2) $\sum_{i\in I} \psi_i(x) \equiv 1$,

(3) there exists a well-spread family $(x_i)_{i\in I}$ *such that* $\operatorname{supp} \psi_i \subset x_i U$.

The construction of BUPU's with respect to arbitrary well-spread sets is standard.

We call $W(B, Y, Q)$ right translation invariant if the right translations R_x (resp. A_x if $B = M$) are bounded operators on $W(L^\infty, Y, Q)$. In [22] the following results were shown.

Theorem 3.1. *The following statements are equivalent:*

(i) $W(L^\infty, Y) = W(L^\infty, Y, Q)$ *is independent of the choice of the neighborhood Q of e (with equivalent norms for different choices).*

(ii) $Y_d = Y_d(X, U)$ *is independent of the choice of the neighborhood U of e (with equivalent norms for different choices).*

(iii) $W(L^\infty, Y) = W(L^\infty, Y, Q)$ *is right translation invariant.*

If one (and hence all) of these conditions are satisfied then:

(a) $W(B, Y) = W(B, Y, Q)$ *is independent of the choice of Q.*

(b) $W(B, Y)$ *is right translation invariant.*

(c) Y_d *and* $W(B, Y)$ *are complete.*

(d) The expression

$$\|F|W(B, Y_d)\| := \|(\|F\chi_{x_iQ}|B\|)_{i\in I}|Y_d(X)\|, \tag{3.4}$$

defines an equivalent quasi-norm on $W(B, Y)$.

Also the left translation invariance is a useful property. For instance, it ensures inclusions into weighted L^∞ spaces, see [22].

Lemma 3.2. *Let $W(L^\infty, Y)$ be left translation invariant. Let $r(x) := \|L_{x^{-1}}|W(L^\infty, Y)\|$ and $\tilde{r}(i) := r(x_i)$. Then*

(a) $Y_d(X)$ *is continuously embedded into* $\ell^\infty_{1/\tilde{r}}$;

(b) $W(L^\infty, Y)$ *is continuously embedded into* $L^\infty_{1/r}$.

The following criterions for left and right translation invariance were provided in [22].

Lemma 3.3. *If Y is left translation invariant then $W(B,Y)$ is left translation invariant and $|||L_y|W(B,Y)||| \leq |||L_y|Y|||$.*

Recall that $\mathcal{G}$ is called an IN group if there exists a compact neighborhood of e such that $xQ = Qx$ for all $x \in \mathcal{G}$.

Lemma 3.4. *Let Y be right translation invariant. Then also $W(B,Y)$ is right translation invariant. Moreover, if $\mathcal{G}$ is an IN group then*

$$|||R_y|W(L^\infty,Y)||| \leq |||R_y|Y||| \quad \text{and} \quad |||A_y|W(M,Y)||| \leq |||R_y|Y|||.$$

We remark that Y does not necessarily need to be translation invariant in order $W(L^\infty,Y)$ to be translation invariant, see [22] for an example. The following result for the involution $^\vee$ will also be useful later on.

Lemma 3.5. *If $\mathcal{G}$ is an IN group then $W(L^\infty,Y^\vee)^\vee = W(L^\infty,Y)$.*

The main ingredient for the coorbit space theory with respect to quasi-Banach spaces will be the following convolution relations for Wiener amalgam spaces.

Theorem 3.6. *Let $0 < p \leq 1$ be such that the quasi-norm of Y satisfies the p-triangle inequality and assume that $W(L^\infty,Y)$ is right translation invariant.*

(a) Set $w(x) := |||A_x|W(M,Y)|||$. Then we have

$$W(M,Y) * W(L^\infty,L^p_w) \hookrightarrow W(L^\infty,Y)$$

with corresponding estimate for the quasi-norms.

(b) Set $v(x) := \Delta(x^{-1})|||R_{x^{-1}}|W(L^\infty,Y)|||$. Then we have

$$W(L^\infty,Y) * W(L^\infty,L^p_v) \hookrightarrow W(L^\infty,Y)$$

with corresponding estimate for the quasi-norms.

Theorem 3.7. *Let w be a submultiplicative weight and $0 < p \leq 1$. Then it holds*

$$W(L^\infty,L^p_w) * W(L^\infty,L^p_{w^*})^\vee \hookrightarrow W(L^\infty,L^p_w).$$

In particular, if $\mathcal{G}$ is an IN-group then $W(L^\infty,L^p_w)*W(L^\infty,L^p_w) \hookrightarrow W(L^\infty,L^p_w)$ with corresponding quasi-norm estimate.

Further, we will need the following maximal function. For some relatively compact neighborhood U of $e \in \mathcal{G}$ and a function G on $\mathcal{G}$ we define the U-oscillation by

$$G_U^{\#}(x) := \sup_{u \in U} |G(ux) - G(x)|.$$

The following lemma on the U-oscillation will be an essential tool for deriving the atomic decomposition for the coorbit spaces defined later on.

Lemma 3.8. *(a) Let $W(L^\infty, Y)$ be left and right translation invariant. If $G \in W(C_0, W(L^\infty, Y)^\vee)^\vee \cap W(C_0, Y)$ then $G_U^{\#} \in W(C_0, Y)$.*

(b) Let w be a submultiplicative weight function and $0 < p < \infty$. Then $G \in W(C_0, W(L^\infty, L_w^p)^\vee)^\vee \cap W(C_0, L_w^p)$ implies

$$\lim_{U \to \{e\}} \|G_U^{\#}|W(L^\infty, L_w^p)\| = 0.$$

(c) If $y \in xU$ then $|L_y G - L_x G| \leq L_y G_U^{\#}$ holds pointwise.

Proof: (a) The left and right translation invariance of $W(L^\infty, Y)$ implies by Theorem 3.1 that $W(C_0, W(L^\infty, Y)^\vee, Q)$ and $W(C_0, Y, Q)$ are independent of the choice of the compact neighborhood Q of $e \in \mathcal{G}$. The control function of $G_U^{\#}$ can be estimated as follows,

$$\begin{aligned} K(G_U^{\#}, Q, L^\infty)(x) &= \sup_{z \in xQ} G_U^{\#}(z) = \sup_{z \in xQ} \sup_{u \in U} |G(uz) - G(z)| \\ &\leq \sup_{z \in xQ} \sup_{u \in U} |G(uz)| + \sup_{z \in xQ} |G(z)| = \sup_{q \in Q} \sup_{u \in U} |G(uxq)| + K(G, Q, C_0)(z). \end{aligned}$$

Clearly, we have $K(G, Q, C_0) \in Y$ by assumption on G. We further compute the function $H(x) := \sup_{q \in Q} \sup_{u \in U} |G(uxq)|$

$$\begin{aligned} H(x) &= \sup_{q \in Q} \|\chi_U(R_{xq}G)\|_\infty = \sup_{q \in Q} \|(R_{(xq)^{-1}}\chi_U)^\vee G^\vee\|_\infty \\ &= \sup_{q \in Q} \|L_{(xq)^{-1}}\chi_{U^{-1}} G^\vee\|_\infty = \sup_{q \in Q} K(G^\vee, U^{-1}, L^\infty)^\vee(xq) \\ &= \|\chi_{xQ} K(G^\vee, U^{-1}, L^\infty)^\vee\|_\infty = K(K(G^\vee, U^{-1}, L^\infty)^\vee, Q, L^\infty)(x). \end{aligned}$$

Thus,

$$\begin{aligned} \|H|Y\| &= \|K(K(G^\vee, U^{-1}, L^\infty)^\vee, Q, L^\infty)|Y\| \\ &\leq C\|G^\vee|W(L^\infty, W(L^\infty, Y)^\vee)\|. \end{aligned}$$

This implies $G_U^{\#} \in W(C_0, Y)$.

(b) By part (a) $G_U^{\#}$ is contained in $W(C_0, L_w^p)$. Let $\epsilon > 0$. Since $U \subset U_0$ implies $G_U^{\#} \leq G_{U_0}^{\#}$ we can find a compact set $V \subset \mathcal{G}$ such that

$$\int_{\mathcal{G} \setminus V} K(G_U^{\#}, Q, L^\infty)(x)^p w(x)^p dx \leq \frac{\epsilon}{2}$$

for all $U \subset U_0$. Since G is uniformly continuous on the compact set VQ we can find a neighborhood $U_1 \subset U_0$ of e such that

$$G^{\#}_{U_1}(x) \leq M := \frac{\epsilon^{1/p}}{(2|V|)^{1/p}\,\nu} \quad \text{for all } x \in VQ$$

with $\nu := \max_{x\in V} w(x)$. This implies

$$K(G^{\#}_{U_1}, Q, L^\infty)(x) = \sup_{z\in xQ} |G^{\#}_{U_1}(z)| \leq M \quad \text{for all } x \in V.$$

Thus, we obtain

$$\int_V K(G^{\#}_{U_1}, Q, L^\infty)(x)^p w(x)^p dx \leq M^p |V| \nu^p = \frac{\epsilon}{2}.$$

Altogether this yields $\|G^{\#}_{U_1}|W(C_0, L^p_w)\|^p \leq \epsilon$.
(c) This is straightforward (see also Lemma 4.6(iii) in [13]). ■

Remark 3.1. *Let $\mathcal{G}$ be an IN-group. Then it follows from Lemma 3.5 that*

$$W(C_0, W(L^\infty, Y)^\vee)^\vee = W(C_0, W(L^\infty, Y)) = W(C_0, Y).$$

The second equality follows from $K(K(F, Q, L^\infty)) \leq K(F, Q^2, L^\infty)$ and the independence of $W(L^\infty, Y, Q)$ of Q. Thus, it suffices to assume $G \in W(C_0, L^p_w)$ in (b) in this case. For general groups, however, such a simplification does not seem possible.

4 Coorbit Spaces

Let π be an irreducible unitary representation of $\mathcal{G}$ on some Hilbert space $\mathcal{H}$. Then the abstract wavelet transform (voice transform) is defined as

$$V_g f(x) := \langle f, \pi(x) g\rangle, \qquad f, g \in \mathcal{H}, x \in \mathcal{G}.$$

The representation π is called square-integrable if there exists a non-zero $g \in \mathcal{H}$ (called admissible) such that $V_g g \in L^2(\mathcal{G})$. Then by a theorem of Duflo and Moore [4] it holds

$$\|V_g f|L^2\| = c_g \|f|\mathcal{H}\| \qquad \text{for all } f \in \mathcal{H} \tag{4.1}$$

with some constant c_g. It can be shown [4] that $c_g = \|Kg|\mathcal{H}\|$ for some uniquely defined self-adjoint, positive and densely defined operator K (possibly unbounded) whose domain $\mathcal{D}(K)$ consists of the admissible vectors. Moreover, if $\mathcal{G}$ is unimodular then K is a multiple of the identity.

As a consequence of (4.1), if g is normalized, i.e., $c_g = \|Kg\| = 1$, we have the reproducing formula

$$V_g f \;=\; V_g f * V_g g. \tag{4.2}$$

In order to introduce the coorbit spaces we first need to extend the definition of the voice transform to a larger space, the "reservoir". To this end let v be some submultiplicative weight function satisfying $v \geq 1$. We define the following class of analyzing vectors

$$\mathbb{A}_v \;:=\; \{g \in \mathcal{H}, V_g g \in L^1_v\}.$$

We assume that $\mathbb{A}_v$ is non-trivial, i.e., π is integrable. This implies that π is also square-integrable. With some fixed $g \in \mathbb{A}_v \setminus \{0\}$ we define

$$\mathcal{H}^1_v \;:=\; \{f \in \mathcal{H}, V_g f \in L^1_v\}$$

with norm $\|f|\mathcal{H}^1_v\| \;:=\; \|V_g f|L^1_v\|$. It can be shown [8, 20] that $\mathcal{H}^1_v$ is a π-invariant Banach space whose definition does not depend on the choice of g. We denote by $(\mathcal{H}^1_v)^\urcorner$ the anti-dual, i.e., the space of all bounded conjugate-linear functionals on $\mathcal{H}^1_v$. An equivalent norm on $(\mathcal{H}^1_v)^\urcorner$ is given by $\|V_g f|L^\infty_{1/v}\|$. Denoting by $\langle \cdot, \cdot \rangle$ also the dual pairing of $(\mathcal{H}^1_v, (\mathcal{H}^1_v)^\urcorner)$ the voice transform extends to $(\mathcal{H}^1_v)^\urcorner$,

$$V_g f(x) \;=\; \langle f, \pi(x) g\rangle, \quad f \in (\mathcal{H}^1_v)^\urcorner, g \in \mathbb{A}_v.$$

Important properties of the voice transform extend to $(\mathcal{H}^1_v)^\urcorner$ as stated in the following lemma, see [8, 9, 20].

Lemma 4.1. *Let $g \in \mathbb{A}_v$ with $\|Kg|\mathcal{H}\| = 1$.*

(a) It holds $V_g(\pi(x)f) = L_x V_g f$ for all $x \in \mathcal{G}$, $f \in (\mathcal{H}^1_v)^\urcorner$.

(b) A bounded net $(f_\alpha) \subset (\mathcal{H}^1_v)^\urcorner$ is weak-$*$ convergent to an element $f \in (\mathcal{H}^1_v)^\urcorner$ if and only if $(V_g f_\alpha)$ converges to $V_g f$ pointwise.

(c) The reproducing formula extends to $(\mathcal{H}^1_v)^\urcorner$, i.e.,

$$V_g f \;=\; V_g f * V_g g \qquad \text{for all } f \in (\mathcal{H}^1_v)^\urcorner.$$

(d) Conversely, if $F \in L^\infty_{1/w}$ satisfies the reproducing formula $F = F * V_g g$ then there exists a unique element $f \in (\mathcal{H}^1_v)^\urcorner$ such that $F = V_g f$.

Let us now define a space of analyzing vectors that allows us to treat also quasi-Banach spaces. For $0 < p \leq 1$ and for some submultiplicative weight function w we define

$$\mathbb{B}^p_w \;:=\; \{g \in \mathcal{D}(K), V_g g \in W(L^\infty, L^p_w)\}.$$

In the sequel we admit only those p and w such that $\mathbb{B}_w^p \neq \{0\}$. Then it follows from the left and right translation invariance of $W(L^\infty, L_w^p)$ and the irreducibility of π that $\mathbb{B}_w^p$ is dense in $\mathcal{H}$. Now we are able to define the coorbit spaces.

Definition 4.1. *Let Y be a solid quasi-Banach space of functions on $\mathcal{G}$ such that $W(L^\infty, Y)$ is left and right translation invariant. Let $0 < p \leq 1$ such that Y has a p-norm and put*

$$w(x) := \max\{|||R_x|W(L^\infty, Y)|||, \Delta(x^{-1})|||R_{x^{-1}}|W(L^\infty, Y)|||\}, \qquad (4.3)$$
$$v(x) := \max\{1, |||L_{x^{-1}}|W(L^\infty, Y)|||\}. \qquad (4.4)$$

We assume that

$$\mathbb{B}(Y) := \mathbb{B}_w^p \cap \mathbb{A}_v \qquad (4.5)$$

is non-trivial. Then for $g \in \mathbb{B}(Y) \setminus \{0\}$ the coorbit space is defined by

$$\mathcal{C}(Y) := \mathsf{Co}W(L^\infty, Y) := \{f \in (\mathcal{H}_v^1)^\urcorner, V_g f \in W(L^\infty, Y)\}$$

with quasi-norm $\|f|\mathcal{C}(Y)\| := \|V_g f|W(L^\infty, Y)\|$.

Let us prove that the reproducing formula extends to $\mathcal{C}(Y)$, and that $\mathcal{C}(Y)$ is complete and independent of the choice of $g \in \mathbb{B}_w^p \setminus \{0\}$.

Proposition 4.2. *Let $g \in \mathbb{B}(Y)$ such that $\|Kg|\mathcal{H}\| = 1$. A function $F \in W(L^\infty, Y)$ is of the form $V_g f$ for some $f \in \mathcal{C}(Y)$ if and only if F satisfies the reproducing formula $F = F * V_g g$.*

Proof: If $f \in \mathcal{C}(Y) \subset (\mathcal{H}_v^1)^\urcorner$ then the reproducing formula $V_g f = V_g f * V_g g$ holds by the reproducing formula for $(\mathcal{H}_v^1)^\urcorner$, see Lemma 4.1(c).
Conversely assume that $F = F * V_g g$ for some $F \in W(L^\infty, Y)$. By Lemma 3.2 $W(L^\infty, Y)$ is embedded into $L_{1/v}^\infty$. Thus $F \in L_{1/v}^\infty$ and by Lemma 4.1(d) it holds $F = V_g f$ for some $f \in (\mathcal{H}_v^1)^\urcorner$, which is then automatically contained in $\mathcal{C}(Y)$ by assumption. ■

Theorem 4.3. *(a) $\mathcal{C}(Y)$ is a quasi-Banach space.*

(b) $\mathcal{C}(Y)$ is independent of the choice of $g \in \mathbb{B}(Y) \setminus \{0\}$.

(c) $\mathcal{C}(Y)$ is independent of the reservoir $(\mathcal{H}_v^1)^\urcorner$ in the following sense: Assume that $\mathbb{S} \subset \mathcal{H}_v^1$ is a locally convex vector space, which is invariant under π, and satisfies $\mathbb{S} \cap \mathbb{B}(Y) \neq \{0\}$. Denote by $\mathbb{S}^\urcorner$ its topological anti-dual. Then for a non-zero vector $g \in \mathbb{B}(Y) \cap \mathbb{S}$ it holds $\mathcal{C}(Y) = \{f \in \mathbb{S}^\urcorner : V_g f \in W(L^\infty, Y)\}$.

Proof: (a) Let $g \in \mathbb{B}_w^p$ such that $\|Kg|\mathcal{H}\| = 1$. Assume $(f_n)_{n\in\mathbb{N}}$ to be a Cauchy sequence in $\mathcal{C}(Y)$. This means that $V_g f_n$ is a Cauchy sequence in $W(L^\infty, Y)$. By completeness of $W(L^\infty, Y)$ there exists a limit $F = \lim_{n\to\infty} V_g f_n$ in $W(L^\infty, Y)$. By Theorem 3.6(b) the definition of the weight w implies that the operator $F \mapsto F * V_g g$ is continuous from $W(L^\infty, Y)$ into $W(L^\infty, Y)$. Hence, we may interchange its application with taking limits, and together with the reproducing formula (Proposition 4.2) this yields

$$F = \lim_{n\to\infty} V_g f_n = \lim_{n\to\infty} V_g f_n * V_g g = F * V_g g.$$

Using Proposition 4.2 once more we see that $F = V_g f$ for some $f \in \mathcal{C}(Y)$. Clearly, $f = \lim_{n\to\infty} f_n$ in $\mathcal{C}(Y)$ and, hence, $\mathcal{C}(Y)$ is complete.
(b) Let $g, g' \in \mathbb{B}_w^p \setminus \{0\}$. Without loss of generality we may assume that g, g' are normalized, i.e., $\|Kg\| = \|Kg'\| = 1$. Choose a vector $h \in \mathbb{B}_w^p$ such that Kh is not orthogonal to Kg and Kg'. It follows from the orthogonality relations that

$$0 \neq \langle Kg', Kh\rangle \langle Kh, Kg\rangle V_g g' = V_{g'} g' * V_h h * V_g g.$$

Since $V_g g^\nabla = V_g g$ and likewise for h and g', and since $w = w^*$ it follows from Theorem 3.7 that $V_g g' \in W(L^\infty, L_w^p)$. The inversion formula for $V_{g'}$ reads $g = \int_{\mathcal{G}} V_{g'} g(y) \pi(y) g' dy$, and one easily deduces

$$V_g f = V_{g'} f * V_g g' \qquad \text{for all } f \in (\mathcal{H}_v^1)^\urcorner. \tag{4.6}$$

By the convolution relation in Theorem 3.6(b) we conclude $V_g f \in W(L^\infty, Y)$ if $V_{g'} f \in W(L^\infty, Y)$. Exchanging the roles of g and g' shows the converse implication.
(c) The π-invariance ensures that the voice transform $V_g f(x) = \langle f, \pi(x) g\rangle$ is well-defined for $f \in \mathbb{S}^\urcorner$. Moreover, it also implies that $\mathbb{S}$ is dense in $\mathcal{H}$ by irreducibility of π. Thus from $\mathbb{S} \subset \mathcal{H}_v^1$, it follows $(\mathcal{H}_v^1)^\urcorner \subset \mathbb{S}^\urcorner$. Hence, the inclusion $\mathcal{C}(Y) \subset \{f \in \mathbb{S}^\urcorner, V_g f \in W(L^\infty, Y)\}$ is clear. Conversely, we have $W(L^\infty, Y) \subset L_{1/v}^\infty$ by Lemma 3.2 and thus $V_g f \in W(L^\infty, Y)$ implies $V_g f \in L_{1/v}^\infty$. This is equivalent to $f \in (\mathcal{H}_v^1)^\urcorner$, and the proof is completed. ■

Remark 4.1. *The assumption that $W(L^\infty, Y)$ is left translation invariant may even be weakened. Analyzing the previous proof it is enough to impose the following conditions: (a) $W(L^\infty, Y)$ is contained in a weighted $L_{1/v}^\infty$ space. (b) There exists a reservoir $\mathbb{S}^\urcorner$ on which the wavelet transform is well-defined such that $\mathcal{H}_{1/v}^\infty := \{f \in \mathbb{S}^\urcorner, V_g f \in L_{1/v}^\infty\}$ satisfies the properties in Lemma 4.1 (for some suitable g).*

Let us give a characterization of the space of analyzing vectors $\mathbb{B}_w^p$. For simplicity we restrict to the case that $\mathcal{G}$ is an IN group although a similar (but more complicated) result can also be formulated in the general case.

Theorem 4.4. *Let $\mathcal{G}$ be an IN-group. Let w be a submultiplicative weight and $0 < p \leq 1$. Define $w^\bullet(x) := \max\{w(x), w(x^{-1})\} \geq 1$. Then it holds*

$$\mathbb{B}_w^p = \mathbb{B}_{w^\bullet}^p = \mathcal{C}(L_{w^\bullet}^p).$$

Proof: Let $g \in \mathbb{B}_w^p$. It follows from $V_g g = V_g g^\nabla$ and Lemma 3.5 that $V_g g \in W(L^\infty, L_{w^\bullet}^p)$, i.e., $g \in \mathbb{B}_{w^\bullet}^p$. Let g' be another element of $\mathbb{B}_w^p = \mathbb{B}_{w^\bullet}^p$. Then the previous proof showed that $V_g g', V_{g'} g \in W(L^\infty, L_{w^\bullet}^p)$ and thus $g, g' \in \mathcal{C}(L_{w^\bullet}^p)$.
Conversely, assume that $g \in \mathcal{C}(L_{w^\bullet}^p)$. Note that $\mathcal{C}(L_{w^\bullet}^p) \subset \mathcal{H} = \mathcal{D}(K)$ (the latter by unimodularity of $\mathcal{G}$) so that voice transforms are well-defined. Let $g' \in \mathbb{B}_{w^\bullet}^p \setminus \{0\}$. Setting $f = g$ in (4.6) shows $V_g g = V_{g'} g * V_g g' = V_{g'} g * (V_{g'} g)^\nabla$. Since both $V_{g'} g$ and $(V_{g'} g)^\nabla$ are contained in $W(L^\infty, L_{w^\bullet}^p)$ (Lemma 3.5) it follows from Theorem 3.7 that $V_g g \in W(L^\infty, L_{w^\bullet}^p)$, i.e., $g \in \mathbb{B}_{w^\bullet}^p$. ■

The following theorem will be useful to prove a weak version of a conjecture in [12, Conjecture 12].

Theorem 4.5. *Let $\mathcal{G}$ be an IN group. Let w be a submultiplicative weight function satisfying $w = w^\vee$ and assume $0 < p \leq 1$. If $V_g f \in W(L^\infty, L_w^p)$ for $f, g \in \mathcal{H}$ then both f and g are contained in $\mathbb{B}_w^p = \mathcal{C}(L_w^p)$.*

Proof: Since $\mathcal{G}$ is unimodular, it holds $\mathcal{H} = \mathcal{D}(K)$. It follows from (4.6) that

$$V_g g = (V_g f)^\nabla * V_g f \quad \text{and} \quad V_f f = V_g f * (V_g f)^\nabla.$$

Since $w(x) = w(x^{-1})$ it follows from Lemma 3.5 that also $(V_g f)^\nabla$ is contained in $W(L^\infty, L_w^p)$. The convolution relation in Theorem 3.7 and Theorem 4.4 finally yields the assertion. ■

5 Discretizations

Our next aim is to derive atomic decompositions for coorbit spaces. The idea is to discretize the reproducing formula. For some suitable function G – later on we will take $G = V_g g$ – we denote the convolution operator by

$$TF := T^G F = F * G = \int_{\mathcal{G}} F(y) L_y G \, dy.$$

For some BUPU $\Psi = (\psi_i)_{i \in I}$ with corresponding well-spread set $X = (x_i)_{i \in I}$ we define the approximation operator

$$T_\Psi F := T_\Psi^{\mathcal{G}} F := \sum_{i \in I} \langle F, \psi_i \rangle L_{x_i} G.$$

We first prove its boundedness.

Proposition 5.1. *Let $X = (x_i)_{i\in I}$ be some well-spread set and $(\psi_i)_{i\in I}$ be a corresponding BUPU of size U. Then the operator $F \mapsto (\langle F, \psi_i\rangle)_{i\in I}$ is continuous from $W(L^\infty, Y)$ into Y_d, i.e., $\|(\langle F, \psi_i\rangle)_{i\in I}|Y_d\| \leq C\|F|W(L^\infty, Y)\|$.*

Proof: Let $F \in W(L^\infty, Y)$ and Q some compact neighborhood of e. Denoting $I_y := \{i \in I : y \in x_i Q\}$ we obtain

$$|\sum_{i\in I}\langle F, \psi_i\rangle \chi_{x_i Q}(y)| \;\leq\; \sum_{i\in I_y}\langle |F|, \psi_i\rangle \;\leq\; \langle L_y \chi_{Q^{-1}U}, |F|\rangle \;=\; |F| * \chi_{U^{-1}Q}(y).$$

The function $\chi_{U^{-1}Q}$ is contained in $W(L^\infty, L^p_w)$ for any p and submultiplicative weight function w. By solidity this yields

$$\begin{aligned}\|(\langle F, \psi_i\rangle)_{i\in I}|Y_d\| \;&\leq\; \|F * \chi_{U^{-1}Q}|Y\| \;\leq\; \|F * \chi_{U^{-1}Q}|W(L^\infty, Y)\|\\ &\leq\; C\|F|W(L^\infty, Y)\|\|\chi_{U^{-1}Q}|W(L^\infty, L^p_w)\|,\end{aligned}$$

where p and w are chosen according to Theorem 3.6(b). ∎

Proposition 5.2. *Let $0 < p \leq 1$ be such that Y has a p-norm. Set $m(x) := \|A_x|W(M, Y)\|$ and $v(x) := \|L_{x^{-1}}|W(L^\infty, Y)\|$. Assume $G \in W(C_0, L^p_m) \cap W(C_0, L^1_v)$. Then the mapping*

$$(\lambda_i)_{i\in I} \mapsto \sum_{i\in I} \lambda_i L_{x_i} G \tag{5.1}$$

defines a bounded operator from Y_d into $W(L^\infty, Y)$. The sum always converges pointwise, and in the norm of $W(L^\infty, Y)$ if the finite sequences are dense in Y_d.

Proof: Let $(\lambda_i)_{i\in I} \in Y_d$. By Lemma 3.8(a) in [9] the sequence $(L_{x_i}G(x))_{i\in I}$ is contained in ℓ^1_v and by the embedding $Y_d \subset \ell^\infty_{1/v}$ (Lemma 3.2) the series in (5.1) converges pointwise. We define the measure $\mu_\Lambda := \sum_{i\in I}\lambda_i\epsilon_{x_i}$ with ϵ_{x_i} being the Dirac measure at x_i. It follows from Theorem 3.1 (in particular (3.4)) that μ_Λ is contained in $W(M, Y)$ with $\|\mu_\Lambda|W(M, Y)\| \leq C\|(\lambda_i)_{i\in I}|Y_d\|$. Observe that $\sum_{i\in I}\lambda_i L_{x_i}G = \mu_\Lambda * G$. Using Theorem 3.6(a) we finally obtain

$$\begin{aligned}&\|\sum_{i\in I}\lambda_i L_{x_i} G|W(L^\infty, Y)\| \;=\; \|\mu_\Lambda * G|W(L^\infty, Y)\|\\ &\leq\; C\|\mu_\Lambda|W(M, Y)\|\,\|G|W(L^\infty, L^p_m)\| \;\leq\; C'\|(\lambda_i)_{i\in I}|Y_d\|\,\|G|W(L^\infty, L^p_m)\|.\end{aligned}$$

This estimation also implies the norm convergence of the series in (5.1) if the finite sequences are dense in Y_d. ∎

Corollary 5.3. *Let Y, p, m and v be as in the previous proposition and assume $G \in W(C_0, L^p_m) \cap W(C_0, L^1_v)$. Then the operator $T_\Psi = T^G_\Psi$ is bounded from $W(L^\infty, Y)$ into $W(L^\infty, Y)$.*

The following theorem will be the key for providing atomic decompositions.

Theorem 5.4. *Let Y have a p-norm for $0 < p \leq 1$ and let w as in Definition 4.1. Suppose $G \in W(C_0, W(L^\infty, L^p_w)^\vee)^\vee \cap W(C_0, L^p_w)$. Further, assume Ψ to be a BUPU of size U. Then it holds*

$$|\!|\!| T^G - T^G_\Psi | W(L^\infty, Y) |\!|\!| \;\leq\; C \| G^\#_U | W(C_0, L^p_w) \|.$$

In particular, we have $|\!|\!| T^G - T^G_\Psi | W(L^\infty, Y) |\!|\!| \to 0$ if $U \to \{e\}$ by Lemma 3.8(b).

Proof: For $F \in W(L^\infty, Y)$ we get

$$\begin{aligned} |TF - T_\Psi F| \;&=\; \left| \sum_{i \in I} \int_{\mathcal{G}} F(y) \psi_i(y)(L_y G - L_{x_i} G) dy \right| \\ &\leq \sum_{i \in I} \int_{\mathcal{G}} |F(y)| \psi_i(y) | L_y G - L_{x_i} G | dy. \end{aligned}$$

Since $\operatorname{supp} \psi_i \in x_i U$ we obtain with Lemma 3.8(c)

$$|TF - T_\Psi F| \;\leq\; \sum_{i \in I} \int_{\mathcal{G}} |F(y)| \psi_i(y) L_y G^\#_U dy \;=\; \int_{\mathcal{G}} |F(y)| L_y G^\#_U dy \;=\; |F| * G^\#_U.$$

By Lemma 3.8(a) we have $G^\#_U \in W(C_0, L^p_w)$. Thus, the convolution relation in Theorem 3.6(b) finally yields

$$\| TF - T_\Psi F | W(L^\infty, Y) \| \;\leq\; C \| F | W(L^\infty, Y) \| \, \| G^\#_U | W(C_0, L^p_w) \|.$$

This gives the estimate for the operator norm. ■

Before stating the general discretization theorem we need to introduce another set of analyzing vectors. Let v, w be the weight functions defined in (4.4),(4.3) and set $\tilde{w}(x) := \max\{w(x), |\!|\!| A_x | W(M, Y) |\!|\!|\}$. Then we define

$$\mathbb{D}(Y) := \mathbb{A}_v \cap \{ g \in \mathcal{H}, V_g g \in W(L^\infty, L^p_{\tilde{w}}) \cap W(L^\infty, W(L^\infty, L^p_w)^\vee)^\vee \}. \quad (5.2)$$

If $\mathcal{G}$ is an IN group and if $\| A_x | W(M, Y) \| \leq C w(x)$ then it holds $\mathbb{D}(Y) = \mathbb{B}(Y)$ by Remark 3.1.

Theorem 5.5. *Let $g \in \mathbb{D}(Y) \setminus \{0\}$. Then there exists a compact neighborhood U of e such that for any U-dense well-spread set $X = (x_i)_{i \in I}$ the family $\{\pi(x_i) g\}_{i \in I}$ forms an atomic decomposition of $\mathcal{C}(Y)$. This means that there exists a sequence $(\lambda_i)_{i \in I}$ of linear bounded functionals on $(\mathcal{H}^1_v)^\urcorner$ (not necessarily unique) such that* *(a)* $f = \sum_{i \in I} \lambda_i(f)\pi(x_i)g$ *for all* $f \in \mathcal{C}(Y)$ *with convergence in the weak-$*$ topology of* $(\mathcal{H}_v^1)^\urcorner$*, and in the quasi-norm topology of* $\mathcal{C}(Y)$ *provided the finite sequences are dense in* Y_d*;*

(b) an element $f \in (\mathcal{H}_v^1)^\urcorner$ *is contained in* $\mathcal{C}(Y)$ *if and only if* $(\lambda_i(f))_{i\in} \in Y_d$ *and*

$$\|(\lambda_i(f))_{i\in I}|Y_d\| \asymp \|f|\mathcal{C}(Y)\| \text{ for all } f \in \mathcal{C}(Y).$$

Proof: Without loss of generality we may assume $\|Kg|\mathcal{H}\| = 1$. Let $G := V_g g$. Then $G \in W(C_0, L_m^p) \cap W(C_0, L_v^1) \cap W(C_0, W(L^\infty, L_w^p)^\vee)^\vee \cap W(C_0, L_w^p)$ with $m(x) := \|A_x|W(M, Y)\|$ by assumption on g. The restriction of the operator T^G to the closed subspace $W(L^\infty, Y) * G$ is the identity by the reproducing formula (Proposition 4.2). By Theorem 5.4 we can find a compact neighborhood U of e such that $\|\!|T^G - T_\Psi^G|W(L^\infty, Y)\|\!| < 1$ for any BUPU Ψ of size U. Hence, T_Ψ is invertible on $W(L^\infty, Y) * G$ by means of the von Neumann series. Let $X = (x_i)_{i\in I}$ be the well-spread set corresponding to Ψ.

If $f \in \mathcal{C}(Y)$ then $V_g f \in W(L^\infty, Y) * G$ by the reproducing formula (Proposition 4.2) and

$$V_g f \;=\; T_\Psi^G (T_\Psi^G)^{-1} V_g f \;=\; \sum_{i\in I} \langle (T_\Psi^G)^{-1} V_g f, \psi_i \rangle L_{x_i} G.$$

We have $L_{x_i} G = L_{x_i} V_g g = V_g(\pi(x_i)g)$, see Lemma 4.1(a). Since V_g is an isometric isomorphism between $\mathcal{C}(Y)$ and $W(L^\infty, Y) * G$ we obtain

$$f \;=\; \sum_{i\in I} \langle (T_\Psi^G)^{-1} V_g f, \psi_i \rangle \pi(x_i) g.$$

Set $\lambda_i(f) := \langle (T_\Psi^G)^{-1} V_g f, \psi_i \rangle$. Clearly, λ_i, $i \in I$, is a linear bounded functional on $\mathcal{C}(Y)$ (and also on $(\mathcal{H}_v^1)^\urcorner$). Since $(T_\Psi^G)^{-1} V_g f \in W(L^\infty, Y) * G \subset W(L^\infty, Y)$ Proposition 5.1 yields

$$\begin{aligned} &\|(\lambda_i(f))_{i\in I}|Y_d\| \;\le\; C\|(T_\Psi^G)^{-1} V_g f|W(L^\infty, Y)\| \\ &\le\; C\|\!|(T_\Psi^G)^{-1}|W(L^\infty, Y) * G\|\!|\, \|V_g f|W(L^\infty, Y)\| \;=\; C\|\!|(T_\Psi^G)^{-1}\|\!| \|f|\mathcal{C}(Y)\|. \end{aligned}$$

Conversely, assume $(\lambda_i)_{i\in I} \in Y_d$. We apply V_g to the series $\sum_{i\in I} \lambda_i \pi(x_i) g$ to obtain (at least formally)

$$F \;:=\; V_g\left(\sum_{i\in I} \lambda_i \pi(x_i) g\right) \;=\; \sum_{i\in I} \lambda_i L_{x_i} G.$$

Since $Y_d \subset \ell_{1/v}^\infty$ and $G \in W(C_0, L_v^1)$ the series defining F converges pointwise and defines a function in $L_{1/v}^\infty$ by Lemma 3.8(a) in [9]. The pointwise convergence of the partial sums of F implies the weak-$*$ convergence of $\sum_{i\in I}\lambda_i\pi(x_i)g = f$, see Lemma 4.1(a),(b). Once f is identified with an element of $(\mathcal{H}_v^1)^\urcorner$ it belongs to $\mathcal{C}(Y)$ by Proposition 5.2, i.e.,

$$\|f|\mathcal{C}(Y)\| \;=\; \|\sum_{i\in I}\lambda_i L_{x_i}G|W(L^\infty,Y)\| \;\leq\; C\|(\lambda_i)_{i\in I}|Y_d\|.$$

Also the type of convergence follows from Proposition 5.2. ■

Remark 5.1. *By using similar techniques one can also provide Banach frames of the form* $\{\pi(x_i)g\}_{i\in I}$ *for* $\mathcal{C}(Y)$, *compare also [13, 21].*

The following auxiliary result will be needed later on.

Theorem 5.6. *Let* $g\in\mathbb{B}(Y)$ *and* X *be some well-spread set. Then*

$$\|(V_gf(x_i))_{i\in I}|Y_d\| \;\leq\; C\|f|\mathcal{C}(Y)\| \qquad \textit{for all } f\in\mathcal{C}(Y).$$

Proof: Let $f\in\mathcal{C}(Y)$. The function V_gf is continuous. Using Theorem 3.1(d) we obtain

$$\begin{aligned}\|(V_gf(x_i))_{i\in}|Y_d\| \;&\leq\; \|(\|V_gf\chi_{x_iU}\|_\infty)_{i\in I}|Y_d\| \;\leq\; C\|V_gf|W(L^\infty,Y)\|\\ &=\; C\|f|\mathcal{C}(Y)\|.\end{aligned}$$

■

In certain situations one might be able to construct certain expansions as in (5.3) below on the level of the Hilbert space $\mathcal{H}$. For instance, the construction of wavelet orthonormal bases of $L^2(\mathbb{R}^d)$ of this type (with $r = 2^d-1$ tensor product wavelets) is well-known. Also tight frames of such kind have been constructed in time-frequency analysis and wavelet analysis. The theorem below states that these expansions extend automatically from $\mathcal{H}$ to general coorbit spaces under certain assumptions. Its proof is a modification of the one in [15].

Theorem 5.7. *Let* $g_r,\gamma_r\in\mathbb{B}(Y)$, $r=1,\ldots,n$, *and* $X=(x_i)_{i\in I}$ *be a well-spread set such that*

$$f \;=\; \sum_{r=1}^{n}\sum_{i\in I}\langle f,\pi(x_i)\gamma_r\rangle\pi(x_i)g_r \quad \textit{for all } f\in\mathcal{H}. \tag{5.3}$$

Then expansion (5.3) extends to all $f\in\mathcal{C}(Y)$ *with norm convergence if the finite sequences are dense in* Y_d *and with weak-$*$ convergence in general. Moreover,* $f\in(\mathcal{H}_v^1)^\urcorner$ *is contained in* $\mathcal{C}(Y)$ *if and only if* $(\langle f,\pi(x_i)\gamma_r\rangle)_{i\in I}$ *is contained in* Y_d *for each* $r=1,\ldots,n$, *and*

$$\|((\langle f,\pi(x_i)\gamma_r\rangle)_{i\in I})_{r=1}^n|\oplus_{r=1}^n Y_d\| \asymp \|f|\mathcal{C}(Y)\| \quad \textit{for all } f\in\mathcal{C}(Y).$$

Proof: We prove the theorem for the case that the finite sequences are dense in Y_d. The general case requires some slight changes.

Let $g \in \mathbb{D}(Y)$ and $Z = (z_j)_{j\in J}$ be a well-spread set as in Theorem 5.5, i.e., such that $\{\pi(z_j)g\}_{j\in J}$ forms an atomic decomposition of $\mathcal{C}(Y)$. Let $f \in \mathcal{C}(Y)$. Then we have the decomposition $f = \sum_{j\in J} \lambda_j(f)\pi(z_j)g$ with $(\lambda_j(f))_{j\in J} \in Y_d(Z)$. Since the finite sequences are dense in Y_d there exists a finite set $N \subset J$ such that $\|\Lambda_N|Y_d(Z)\| < \epsilon$ where $(\Lambda_N)_j = \lambda_j$ if $j \notin N$ and $(\Lambda_N)_j = 0$ otherwise. The element $f_N = \sum_{j\in N} \lambda_j \pi(z_j)g$ is contained in $\mathcal{H} \cap \mathcal{C}(Y)$ and satisfies

$$\|f - f_N|\mathcal{C}(Y)\| \;\leq\; C\|\Lambda_N|Y_d(Z)\| \;\leq\; C\epsilon.$$

By assumption f_N has the expansion

$$f_N \;=\; \sum_{r=1}^{n}\sum_{i\in I}\langle f_N, \pi(x_i)\gamma_r\rangle \pi(x_i)g_r.$$

Theorem 5.6 asserts that the sequence $(\langle f_N, \pi(x_i)\gamma_r\rangle)_{i\in I} = (V_{\gamma_r} f_N(x_i))_{i\in I}$ is contained in $Y_d(X)$ for all $r = 1,\ldots,n$. As above there exist finite sets N_r, $r = 1,\ldots,n$, such that $\|\kappa^{(r)}|Y_d(X)\| \leq \epsilon$ where $\kappa_i^{(r)} = \langle f_N, \pi(x_i)\gamma_r\rangle$ if $i \notin N_r$ and $\kappa_i^{(r)} = 0$ otherwise. Then $f_1 := \sum_{r=1}^{n}\sum_{i\in N_r}\langle f_N, \pi(x_i)\gamma_r\rangle\pi(x_i)g_r$ satisfies

$$\|f_N - f_1|\mathcal{C}(Y)\| \leq C'\sum_{r=1}^{n}\|\kappa^{(r)}|Y_d(X)\| \leq Cr\epsilon.$$

It follows from Proposition 5.2 that

$$\begin{aligned}\|\sum_{r=1}^{n}\sum_{i\in I}\lambda_{i,r}\pi(x_i)g_r|\mathcal{C}(Y)\|^p \;&\leq\; \sum_{r=1}^{n} C_r\|\sum_{i\in I}\lambda_{i,r}L_{x_i}V_{g_r}g_r|W(L^\infty, Y)\|^p \\ \leq\; C\sum_{r=1}^{n}\|(\lambda_{i,r})_{i\in I}|Y_d(X)\|^p \;&\leq\; C'\|((\lambda_{i,r})_{i\in I})_{r=1}^{n}|\oplus_{r=1}^{n} Y_d(X)\|^p\end{aligned}$$

for all $(\lambda_{i,r}) \in \oplus_{r=1}^{n} Y_d(X)$. With $f_2 := \sum_{r=1}^{n}\sum_{i\in N_r}\langle f, \pi(x_i)\gamma_r\rangle\pi(x_i)g_r$ this yields using Theorem 5.6

$$\begin{aligned}\|f_1 - f_2|\mathcal{C}(Y)\| \;&\leq\; C\|((\langle f_N - f, \pi(x_i)\gamma_r\rangle)_{i\in I})_{r=1}^{n}|\oplus_{r=1}^{n} Y_d\| \\ &\leq\; C''\|f_N - f|\mathcal{C}(Y)\| \;\leq\; C''\epsilon.\end{aligned}$$

Altogether we get

$$\begin{aligned}\|f - f_2|\mathcal{C}(Y)\| \;&\leq\; \|f - f_N|\mathcal{C}(Y)\| + \|f_N - f_1|\mathcal{C}(Y)\| + \|f_1 - f_2|\mathcal{C}(Y)\| \\ &\leq\; (C + C'r + C'')\epsilon.\end{aligned}$$

Since ϵ can be chosen arbitrarily small we deduce that

$$f = \sum_{r=1}^{n} \sum_{i \in I} \langle f, \pi(x_i)\gamma_r \rangle \pi(x_i) g_r$$

for all $f \in \mathcal{C}(Y)$ with quasi-norm convergence. Moreover, it follows from Theorem 5.6 and Proposition 5.2 that

$$\begin{aligned} \|f|\mathcal{C}(Y)\| &= \| \sum_{r=1}^{n} \sum_{i \in I} \langle f, \pi(x_i)\gamma_r \rangle \pi(x_i) g_r |\mathcal{C}(Y)\| \\ &\leq C(\sum_{r=1}^{n} \|(\langle f, \pi(x_i)\gamma_r \rangle)_{i \in I} | Y_d(X)\|^p)^{1/p} \\ &\leq C(\sum_{r=1}^{n} \|V_{\gamma_r} f | W(L^\infty, Y)\|^p)^{1/p} \leq C' \|f|\mathcal{C}(Y)\|. \end{aligned}$$

This concludes the proof. ■

6 Characterizations of $\mathcal{C}(Y)$ via Y

The original definition of the coorbit spaces by Feichtinger and Gröchenig involves Y rather than $W(L^\infty, Y)$. It is interesting to investigate what happens if we replace $W(L^\infty, Y)$ by Y in our more general case. In order to distinguish clearly between the two spaces let us denote $\mathsf{Co}Y = \{f \in (\mathcal{H}_v^1)^\urcorner, V_g f \in Y\}$ with natural norm $\|f|\mathsf{Co}Y\| = \|V_g f|Y\|$, and $\mathsf{Co}W(L^\infty, Y) = \mathcal{C}(Y)$ as usual. It was already proven in [10] that in the classical Banach space case both spaces coincide:

Theorem 6.1. *(Theorem 8.3 in [10]) Let Y be a solid Banach space of functions on $\mathcal{G}$ that is left and right translation invariant and continuously embedded into $L^1_{loc}(\mathcal{G})$. Then it holds $\mathsf{Co}Y = \mathsf{Co}W(L^\infty, Y)$ with equivalent norms.*

In the general case of quasi-Banach spaces at least the inclusion $\mathsf{Co}W(L^\infty, Y) \subset \mathsf{Co}Y$ holds since $W(L^\infty, Y) \subset Y$. However, it seems doubtful that we can state results on the converse inclusion in the general abstract case. Indeed, in order to come up with an abstract result we would need a convolution relation of the type $Y * W(L^\infty, L^p_w) \hookrightarrow W(L^\infty, Y)$ (see [9] for the Banach space case). However, such a relation does not hold for general quasi-Banach spaces (consider $Y = L^p(\mathcal{G})$, $0 < p < 1$, for a non-discrete group $\mathcal{G}$).

Moreover, it is even not clear whether $\mathsf{Co}Y$ is a complete space. Indeed, in the proof of completeness of $\mathcal{C}(Y)$ (and of $\mathsf{Co}Y$ in the case of Banach spaces

Y) one makes heavy use of the reproducing formula together with the fact that convolution from the right with a suitable G is a bounded operator from $W(L^\infty, Y)$ into itself.

In special cases, however, one might be able to prove that $\|V_g f|W(L^\infty, Y)\| \leq C\|V_g f|Y\|$ for a very specific choice of g, by using methods that are not available in the abstract setting (like analyticity properties for instance), see e.g. Section 7. Then one may extend this inequality to more general analyzing vectors g as shown by the next result.

Theorem 6.2. *Let Y be a right translation solid p-normed quasi-Banach space of functions on $\mathcal{G}$ such that $W(L^\infty, Y)$ is left translation invariant. Set $\nu(x) := |\!|\!| R_x|Y |\!|\!|$. Let $\mathbb{S} \subset \mathcal{H}_v^1$ be a π-invariant locally convex vector space (with $v(x) := \max\{1, |\!|\!| L_{x^{-1}}|W(L^\infty, Y)|\!|\!|\}$, see (4.4)). Assume that there exists a non-zero vector $g_0 \in \mathbb{B}(Y) \cap \mathcal{C}(L^p_\nu) \cap \mathbb{S}$ and a constant $C > 0$ such that*

$$\|V_{g_0} f|W(L^\infty, Y)\| \leq C\|V_{g_0} f|Y\|$$

for all $f \in \mathbb{S}^\urcorner \supset (\mathcal{H}_v^1)^\urcorner$ (with the understanding that $V_{g_0} f \in W(L^\infty, Y)$ if $V_{g_0} f \in Y$). Let $g \in \mathbb{B}(Y) \cap \mathbb{D}(L^p_\nu) \cap \mathbb{S} \setminus \{0\}$. Then it holds

$$\|V_g f|W(L^\infty, Y)\| \asymp \|V_g f|Y\|$$

for all $f \in \mathbb{S}^\urcorner \supset (\mathcal{H}_v^1)^\urcorner$ and $\mathsf{Co}W(L^\infty, Y) = \mathsf{Co}Y = \{f \in \mathbb{S}^\urcorner, V_g f \in Y\}$. In particular, $\mathsf{Co}Y$ is complete with the quasi-norm $\|V_g f|Y\|$ and independent of the choice of $g \in \mathbb{B}(Y) \cap \mathbb{D}(L^p_\nu) \cap \mathcal{H}_v^1 \setminus \{0\}$.

Proof: Since $\mathcal{C}(Y)$ is independent of the choice of $g \in \mathbb{B}(Y) \setminus \{0\}$ and of the reservoir $\mathbb{S}$ (Theorem 4.3) we have

$$\|V_g f|W(L^\infty, Y)\| \leq C\|V_{g_0} f|W(L^\infty, Y)\|.$$

for all $f \in \mathbb{S}^\urcorner$. Thus, it remains to prove that $\|V_{g_0} f|Y\| \leq C\|V_g f|Y\|$ for all $f \in \mathbb{S}^\urcorner$. By the assumptions on g it follows from Theorem 5.5 that g_0 has a decomposition

$$g_0 = \sum_{i \in I} \lambda_i(g_0)\pi(x_i)g$$

with $(\lambda_i(g_0))_{i\in I} \in \ell^p_\nu = (L^p_\nu)_d$ and $\|(\lambda_i(g_0))_{i\in I}|\ell^p_\nu\| \asymp \|g_0|\mathcal{C}(L^p_\nu)\|$. Hence, we obtain

$$V_{g_0} f(x) = \langle f, \pi(x)g_0\rangle = \langle f, \pi(x)\sum_{i\in I} \lambda_i(g_0)\pi(x_i)g\rangle = \sum_{i\in I} \overline{\lambda_i(g_0)} R_{x_i} V_g f(x).$$

This yields

$$\begin{aligned} \|V_{g_0} f|Y\|^p &= \|\sum_{i\in I} \overline{\lambda_i(g_0)} R_{x_i} V_g f|Y\|^p \leq \sum_{i\in I} |\lambda_i(g_0)|^p |\!|\!| R_{x_i}|Y|\!|\!|^p \|V_g f|Y\|^p \\ &\leq C\|g_0|\mathcal{C}(L^p_\nu)\|^p \, \|V_g f|Y\|^p \end{aligned}$$

for all $f \in \mathbb{S}^\urcorner$. The reverse inequality $\|V_g f|Y\| \leq \|V_g f|W(L^\infty, Y)\|$ is clear. ■

7 Nonlinear Approximation

Let us now discuss non-linear approximation. Let $(x_i)_{i\in I}$ be some well-spread set and g such that $\{\pi(x_i)g\}_{i\in I}$ forms an atomic decomposition of the coorbit space we want to consider. We denote by

$$\sigma_n(f,\mathcal{C}(Y)) := \inf_{N\subset I,\#N\leq n} \|f - \sum_{i\in N} \lambda_i \pi(x_i)g|\mathcal{C}(Y)\|$$

the error of best n-term approximation in $\mathcal{C}(Y)$. Hereby, the infimum is also taken over all possible choices of coefficients λ_i. Our task is to find a class of elements for which this error has a certain decay when n tends to ∞.

To this end we consider coorbits with respect to Lorentz spaces. For some measurable function F on $\mathcal{G}$ let $\lambda_F(s) = |\{x : |F(x)| > s\}|$ denote its distribution function and $F^*(t) = \inf\{s : \lambda_F(s) \leq t\}$ its non-increasing rearrangement. For $0 < p, q < \infty$ we let

$$\|F\|^*_{p,q} := \left(\frac{q}{p}\int_0^\infty F^*(t)^q t^{q/p}\frac{dt}{t}\right)^{1/q}$$

$$\text{and} \qquad \|F\|^*_{p,\infty} := \sup_{t>0} t^{1/p}F^*(t). \tag{7.1}$$

The Lorentz-space $L(p,q)$ is defined as the collection of all F such that the quantity above is finite. It is well-known that $L(p,p) = L^p$. Most interesting for our purposes is $L(p,\infty)$, which is also called weak-L^p. It holds $L^p \subset L(p,\infty)$. Sometimes it is useful to introduce another quasi-norm on $L(p,q)$. For some r satisfying $0 < r \leq 1$, $r < p$ and $r \leq q$ we let

$$F^{**}(t,r) := \sup_{t<\mu(E)<\infty} \left(\frac{1}{\mu(E)}\int_E |F(x)|^r dx\right)^{1/r}, \tag{7.2}$$

the supremum taken over measurable subsets of $\mathcal{G}$ with the stated property. We define

$$\|F\|^{(r)}_{p,q} := \left(\frac{q}{p}\int_0^\infty F^{**}(t,r)^q t^{q/p}\frac{dt}{t}\right)^{1/q}$$

with modification for $q = \infty$ as in (7.1). It can be easily seen from (7.2) that $\|\cdot\|^{(r)}_{p,q}$ is a quasi-norm. If $r = 1$ (implying $p > 1$) and $q \geq 1$ then it is even a norm, and if $r < 1$ and $q \geq 1$ then $\|\cdot\|^{(r)}_{p,q}$ is an r-norm. Furthermore, if it can be shown [18] that

$$\|F\|^*_{p,q} \leq \|F\|^{(r)}_{p,q} \leq \left(\frac{p}{p-r}\right)^{1/r} \|F\|^*_{p,q}.$$

In particular, also $\|F\|^*_{p,q}$ is a quasi-norm. Moreover, if $p > 1$ and $q \geq 1$ then $L(p,q)$ with the equivalent norm $\|\cdot\|^{(1)}_{p,q}$ is a Banach space. In the general case, $L(p,q)$ is only a quasi-Banach space. Indeed, it is known that $L(1,q)$ is not normable for $q>1$ (except for the trivial case that the underlying group is finite).

By the properties of the Haar-measure it is easily seen that all spaces $L(p,q)$ are left and right translation invariant. Thus, if m is a moderate function then also

$$L_m(p,q) \;=\; \{F \text{ measurable}, Fm \in L(p,q)\}$$

with the quasi-norm $\|F|L_m(p,q)\| := \|Fm\|^*_{p,q}$ is left and right translation invariant. In particular, the Wiener amalgam spaces $W(L^\infty, L_m(p,q))$ are well-defined. Further, if $\mathbb{B}(L_m(p,q))$, see (4.5), is non-trivial then also the coorbit space $\mathcal{C}(L_m(p,q))$ is well-defined.

It is not difficult to see that the sequence space $(L_m(p,q))_d(X)$ associated to a well-spread set $X=(x_i)_{i\in I}$ coincides with a Lorentz space $\ell_m(p,q)$ on the index set I. In particular, an equivalent quasi-norm on $(L_m(p,\infty))_d(X)$ is given by

$$\|(\lambda_i)_{i\in I}\|^*_{p,\infty} \;=\; \sup_{n\in\mathbb{N}} n^{1/p}(\lambda m)^*(n) \tag{7.3}$$

where $(\lambda m)^*$ denotes the non-increasing rearrangement of the sequence $(\lambda_i m(x_i))_{i\in I}$.

Theorem 7.1. *Let m be some w-moderate weight function on $\mathcal{G}$, let $0<p<q\leq\infty$ and define $\alpha=1/p-1/q>0$. Let $(x_i)_{i\in I}$ be some well-spread set and $g\in\mathbb{D}(L_m(p,\infty))\subset\mathbb{D}(L^q_m)$ such that $\{\pi(x_i)g\}_{i\in I}$ forms an atomic decomposition simultaneously of $\mathcal{C}(L_m(p,\infty))$ and $\mathcal{C}(L^q_m)$ (according to Theorem 5.5). Then for all $f\in\mathcal{C}(L_m(p,\infty))$ it holds*

$$\sigma_n(f,\mathcal{C}(L^q_m)) \;\leq\; C\|f|\mathcal{C}(L_m(p,\infty))\|\, n^{-\alpha}. \tag{7.4}$$

__**Proof:**__ Let $f=\sum_{i\in I}\lambda_i(f)\pi(x_i)g$ be an expansion of $f\in\mathcal{C}(L_m(p,\infty))$ in terms of the atomic decomposition. By Theorem 5.5 it holds $(\lambda m)^*_k \leq C\|f|\mathcal{C}(L_m(p,\infty))\|k^{-1/p}$. Let $\tau:\mathbb{N}\to I$ be a bijection that realizes the non-increasing rearrangement, i.e., $\lambda_{\tau(k)}m(x_{\tau(k)})=(\lambda m)^*_k$. Moreover, $(L^q_m)_d=\ell^q_m(I)$, and $\|(\lambda_i(f))_{i\in I}|\ell^q_m(I)\|$ forms an equivalent norm on $\mathcal{C}(L^q_v)$ once again by Theorem 5.5. We obtain

$$\begin{aligned}
&\sigma_n(f,\mathcal{C}(L^q_m)) \;\leq\; \|f-\sum_{k=1}^{n}\lambda_{\tau(s)}\pi(x_{\tau(s)})g|\mathcal{C}(L^q_m)\| \\
&=\; \|\sum_{k=n+1}^{\infty}\lambda_{\tau(k)}\pi(x_{\tau(k)})g|\mathcal{C}(L^q_m)\| \;\leq\; C\left(\sum_{k=n+1}^{\infty}((\lambda m)^*_k)^q\right)^{1/q} \\
&\leq\; C\|f|\mathcal{C}(L_m(p,\infty))\|\left(\sum_{k=n+1} k^{-q/p}\right)^{1/q} \;\leq\; C\|f|\mathcal{C}(L_m(p,\infty))\|(n^{-q/p+1})^{1/q} \\
&=\; C\|f|\mathcal{C}(L_m(p,\infty))\|n^{-\alpha}.
\end{aligned}$$

This completes the proof. ■

Remark 7.1. *(a) The obvious embedding $\mathcal{C}(L^p_m) \subset \mathcal{C}(L_m(p,\infty))$ implies that we also have $\sigma_n(f, \mathcal{C}(L^q_m)) \le Cn^{1/q-1/p}$ for all $f \in \mathcal{C}(L^p_m)$ if $p < q$. However, in this situation one may even prove a slightly faster decay of $\sigma_n(f, \mathcal{C}(L^p_m))$, i.e, $o(n^{1/q-1/p})$ instead of $\mathcal{O}(n^{1/q-1/p})$, with methods similar as in [16] for instance.*

(b) In order to have a very fast decay of $\sigma_n(f, \mathcal{C}(L^q_m))$ one obviously has to take p very small in the Theorem above, in particular, $p \le 1$. Clearly, $\mathcal{C}(L_m(p,\infty))$ is no longer a Banach space in this case, but only a quasi-Banach space. So it is very natural to treat also the case of quasi-Banach spaces when dealing with problems in non-linear approximation. This was actually one of the motivations for this paper.

(c) The most interesting case appears when taking $q = 2$ and $m \equiv 1$ since $\mathcal{C}(L^2) = \mathcal{H}$. So for all $f \in \mathcal{C}(L(p,\infty))$, $p < 2$, we have

$$\sigma_n(f, \mathcal{H}) \;\le\; C\|f|\mathcal{C}(L(p,\infty))\|\, n^{-1/p+1/2}.$$

8 Modulation spaces

Let $\mathbb{H}_d := \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{T}$ denote the (reduced) Heisenberg group with group law

$$(x,\omega,\tau)(x',\omega',\tau') \;=\; (x+x', \omega+\omega', \tau\tau' e^{\pi i(x'\cdot\omega - x\cdot\omega')}).$$

It is unimodular and has Haar measure

$$\int_{\mathbb{H}_d} f(h)dh = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_0^1 f(x,\omega,e^{2\pi i t})dt d\omega dx.$$

We denote by

$$T_x f(t) \;:=\; f(t-x), \quad \text{and} \quad M_\omega f(t) \;=\; e^{2\pi i\omega\cdot t}f(t), \quad x,\omega,t \in \mathbb{R}^d,$$

the translation and modulation operator on $L^2(\mathbb{R}^d)$. Then the Schrödinger representation ρ is defined by

$$\rho(x,\omega,\tau) \;:=\; \tau e^{\pi i x\cdot\omega}T_x M_\omega \;=\; \tau e^{-\pi i x\cdot\omega}M_\omega T_x.$$

It is well-known that this is an irreducible unitary and square-integrable representation of $\mathbb{H}_d$. The corresponding voice transform is essentially the short time Fourier transform:

$$V_g f(x,\omega,\tau) \;=\; \langle f, \rho(x,\omega,\tau)g\rangle_{L^2(\mathbb{R}^d)} \;=\; \overline{\tau}\int_{\mathbb{R}^d} f(t)\overline{e^{-\pi i x\cdot\omega}M_\omega T_x g(t)}dt$$
$$=\; \overline{\tau}e^{\pi i x\cdot\omega}\int_{\mathbb{R}^d} f(t)\overline{g(t-x)}e^{-2\pi i t\cdot\omega}dt \;=\; \overline{\tau}e^{\pi i x\cdot\omega}\,\mathrm{STFT}_g\, f(x,\omega). \qquad (8.1)$$

Let us now introduce the modulation spaces on $\mathbb{R}^d$. We consider nonnegative continuous weight functions m on $\mathbb{R}^d \times \mathbb{R}^d$ that satisfy

$$m(x+y,\omega+\xi) \le C(1+|x|^2+|\omega|^2)^{a/2} m(y,\xi), \quad (x,\omega),(y,\xi) \in \mathbb{R}^d \times \mathbb{R}^d.$$

for some constants $C > 0, a \ge 0$. This means that m is a w-moderate function for $w(x,\omega) = (1+|x|^2+|\omega|^2)^{a/2}$, see also [14, Chapter 11.1]. Additionally, we require m to be symmetric, i.e., $m(-x,-\omega) = m(x,\omega)$. A typical choice is $m_s(x,\omega) = (1+|\omega|)^s, s \in \mathbb{R}$. For $0 < p,q \le \infty$ and m as above we introduce $L^{p,q}_m := L^{p,q}_m(\mathbb{R}^{2d}) := \{F \text{ measurable}, \|F|L^{p,q}_m\| < \infty\}$ with quasi-norm

$$\|F|L^{p,q}_m\| := \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|F(x,\omega)|^p m(x,\omega)^p dx\right)^{q/p} d\omega\right)^{1/q}.$$

This expression is an r-norm with $r := \min\{1,p,q\}$.

Let g be some non-zero Schwartz function on $\mathbb{R}^d$. The short time Fourier transform STFT_g extends to the space $\mathcal{S}'(\mathbb{R}^d)$ of tempered distributions in a natural way. Given $0 < p,q \le \infty$ and m as above the modulation space is defined as

$$M^{p,q}_m := \{f \in \mathcal{S}'(\mathbb{R}^d), \|\operatorname{STFT}_g f|L^{p,q}_m\| < \infty\}$$

with quasi-norm $\|f|M^{p,q}_m\| = \|\operatorname{STFT}_g f|L^{p,q}_m\|$. Since by (8.1) $|V_g f(x,\omega,\tau)| = |\operatorname{STFT}_g f(x,\omega)|$ we can identify the modulation spaces with coorbit spaces,

$$M^{p,q}_m(\mathbb{R}^d) = \mathsf{Co}L^{p,q}_m(\mathbb{H}_d) = \{f \in \mathcal{S}', V_g f \in L^{p,q}_m\},$$

where m and $L^{p,q}_m$ are extended to $\mathbb{H}_d$ in an obvious way, e.g. $m(x,\omega,\tau) = m(x,\omega)$. However, at the moment we do not know yet whether $\mathsf{Co}L^{p,q}_m$ coincides with

$$\mathcal{C}(L^{p,q}_m) = \{f \in \mathcal{S}'(\mathbb{R}^d), V_g f \in W(L^\infty, L^{p,q}_m)\}$$

if $p < 1$ or $q < 1$. It is even not clear yet whether $M^{p,q}_m$ is complete. We will use Theorem 6.2 and a result from [12] to clarify this problem. Let us first investigate the spaces $\mathbb{B}(L^{p,q}_m)$ and $\mathbb{D}(L^{p,q}_m)$, see Definition 4.1 and (5.2). One easily shows [20, Lemma 4.7.1] (using the symmetry of w) that $\|L_{(x,\omega,\tau)}|L^{p,q}_m\| \le w(x,\omega)$ and $\|R_{(x,\omega,\tau)}|L^{p,q}_m\| \le w(x,\omega)$. Since $\mathbb{H}_d$ is an IN group Lemma 3.4 yields

$$\|R_{(x,\omega,\tau)}|W(L^\infty, L^{p,q}_m)\| \le w(x,\omega) \quad \text{and} \quad \|A_{(x,\omega,\tau)}|W(M, L^{p,q}_m)\| \le w(x,\omega).$$

Further, $\|L_{(x,\omega,\tau)}|W(L^\infty, L^{p,q}_m)\| \le w(x,\omega)$ by Lemma 3.3. Thus, using Remark 3.1 we conclude

$$\mathbb{B}^r_w \subset \mathbb{B}(L^{p,q}_m) \quad \text{and} \quad \mathbb{B}^r_w \subset \mathbb{D}(L^{p,q}_m) \quad \text{with } r := \min\{1,p,q\}.$$

Moreover, Theorem 4.4 yields

$$\mathbb{B}_w^r = \mathcal{C}(L_w^r).$$

Let $g_0(t) = e^{-\pi|t|^2}$ be a Gaussian. Using the relation of STFT_{g_0} to the Bargmann transform Galperin and Samarah proved that

$$\|V_{g_0} f | W(L^\infty, L_m^{p,q})\| \le C \|V_{g_0} f | L_m^{p,q}\|$$

for all $f \in M_m^{p,q}$ [12, Lemma 3.2]. Moreover, it is clear that $g_0 \in \mathcal{C}(L_w^r) \subset \mathbb{B}(L_m^{p,q}) \cap \mathcal{C}(L_\nu^p)$ and $\mathcal{S} \subset \mathcal{C}(L_w^r) \subset \mathbb{B}(L_m^{p,q}) \cap \mathbb{D}(L_\nu^p)$ with $\nu(x,\omega) := \|\!| R_{(x,\omega)} | W(L^\infty, L_m^{p,q})\|\!|$. Thus, by Theorem 6.2

$$\mathcal{C}(L_m^{p,q}) = M_m^{p,q},$$

and the latter is complete. It seems that the completeness of $M_m^{p,q}$ for $p < 1$ or $q < 1$ was not stated in [12] or elsewhere in the literature although its proof is somehow hidden in [12].

The abstract discretization Theorem 5.5 yields the following result for atomic decompositions of modulation spaces.

Theorem 8.1. *Let $0 < p_0 \le 1$ and w be some symmetric submultiplicative weight function on $\mathbb{R}^d \times \mathbb{R}^d$ with polynomial growth. Assume $g \in M_w^{p_0}$. Then there exist constants $a, b > 0$ such that*

$$\{M_{bj} T_{ak} g : k, j \in \mathbb{Z}^d\}$$

forms an atomic decomposition for all modulation spaces $M_m^{p,q}$ with $p_0 \le p, q \le \infty$ and m being a w-moderate weight. This means that there exist functionals $\lambda_{k,j}$, $k, j \in \mathbb{Z}^d$ on $M_{1/w}^\infty$ ($\subset \mathcal{S}'$) such that

- *(a) any $f \in M_m^{p,q}$ has the series expansion $f = \sum_{k,j \in \mathbb{Z}^d} \lambda_{k,j}(f) M_{bj} T_{ak} g$;*
- *(b) a distribution $f \in M_{1/w}^\infty$ belongs to $M_m^{p,q}$ if and only if $(\lambda_{k,j}(f))_{k,j \in \mathbb{Z}^d}$ belongs to $\ell_m^{p,q}(\mathbb{Z}^{2d})$, and we have the quasi-norm equivalence*

$$\|f | M_m^{p,q}\| \asymp \left(\sum_{j \in \mathbb{Z}^d} \left(\sum_{k \in \mathbb{Z}^d} |\lambda_{k,j}(f)|^p m(ak, bj)^p \right)^{q/p} \right)^{1/q}$$
$$=: \|(\lambda_{k,j}(f)) | \ell_m^{p,q}(\mathbb{Z}^{2d})\|.$$

We remark that the abstract Theorem 5.5 allows to extend the previous result also to irregular Gabor frames on $M_m^{p,q}$.

Theorem 8.1 indicates that the modulation spaces $M_w^{p_0}$ with $0 < p_0 \le 1$ are the correct window classes for time-frequency analysis on $M_m^{p,q}$. This was already conjectured in [12]. Galperin and Samarah also conjectured that whenever $\mathrm{STFT}_g f \in L_v^p$ then $f \in M_v^p$ and $g \in M_v^p$ [12, Conjecture 12]. Theorem 4.5 leads to a weak version of this conjecture.

Theorem 8.2. *Let $f, g \in L^2(\mathbb{R}^d)$ and $0 < p \leq 1$. Assume v to be a symmetric submultiplicative weight function. If $\mathrm{STFT}_g f \in W(L^\infty, L^p_v)$ then $g \in M^p_v$ and $f \in M^p_v$.*

The remaining question is whether $V_g f \in L^p_v$ already implies $V_g f \in W(L^\infty, L^p_v)$. Let us also apply Theorem 5.7 to our situation.

Theorem 8.3. *Let $g \in \mathcal{S}(\mathbb{R}^d)$ and $a, b > 0$ such that*

$$\{M_{bj}T_{ak}g : j, k \in \mathbb{Z}^d\} \tag{8.2}$$

forms a Gabor frame for $L^2(\mathbb{R}^d)$. Then its canonical dual γ is also contained in $\mathcal{S}(\mathbb{R}^d)$, and any $f \in M^{p,q}_m$, $0 < p, q \leq \infty$, has a decomposition

$$f = \sum_{j,k\in\mathbb{Z}^d} \langle f, M_{bj}T_{ak}g\rangle M_{bj}T_{ak}\gamma$$

with $\|f|M^{p,q}_m\| \asymp \|(\langle f, M_{bj}T_{ak}g\rangle)_{j,k\in\mathbb{Z}^d}|\ell^{p,q}_m(\mathbb{Z}^{2d})\|$.

Proof: Since (8.2) forms a Gabor frame with dual window γ any $f \in L^2(\mathbb{R}^d)$ has a decomposition

$$f = \sum_{j,k\in\mathbb{Z}^d} \langle f, M_{bj}T_{ak}g\rangle M_{bj}T_{ak}\gamma.$$

It was shown in [14, Corollary 13.5.4] that also the dual window γ is contained in $\mathcal{S}(\mathbb{R}^d)$. Since $\mathcal{S}(\mathbb{R}^d) \subset M^{p,q}_m$ for all $0 < p, q \leq \infty$ and all w-moderate weights m with w having polynomial growth we have $g, \gamma \in \mathbb{B}(L^{p,q}_m) = M^r_w$ with $r = \min\{1, p, q\}$. Clearly, the set $\{(ak, bj), k, j \in \mathbb{Z}^d\}$ is well-spread in $\mathbb{H}^d$. Thus, the assertion follows from Theorem 5.7. ■

Depending on p, q and m one may also formulate weaker conditions on g according to the abstract Theorem 5.7 such that the previous theorem still holds.

Of course, one can also apply Theorem 7.1 to non-linear approximation with Gabor frames. We leave this straightforward task to the interested reader. We only remark that apart from a small note in [17] modulation spaces with respect to Lorentz-spaces $L(p, q)$ did not appear in the literature in explicit form before.

Acknowledgement

Major parts of this paper were developed during a stay at the Mathematical Institute of the University of Wrocław, Poland. The author would like to thank its members for their warm hospitality. In particular, he expresses his gratitude to M. Paluszyński and R. Szwarc for interesting discussions on the subject. The author was supported by the European Union's Human Potential Programme under contracts HPRN-CT-2001-00273 (HARP) and HPRN-CT-2002-00285 (HASSIP).

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