

SPARSE RECOVERY FOR SPHERICAL HARMONIC EXPANSIONS

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ABSTRACT

We show that sparse spherical harmonic expansions can be recovered from a small number of randomly chosen samples on the sphere using ℓ_1 -minimization. To this end, we establish the restricted isometry property of an associated preconditioned random measurement matrix.

1. INTRODUCTION

Compressed sensing has triggered significant research activity in recent years. It predicts that sparse signals can be recovered from what was previously believed to be highly incomplete information. In this work, we show that functions on the sphere $S^2 = \{\mathbf{x} \in \mathbb{R}^3, \|\mathbf{x}\|_2 = 1\}$ that have a sparse or compressible representation in the spherical harmonic basis can be recovered from a number of samples that scales (essentially) linearly with the sparsity level. This can be viewed as an extension of existing results [7, 8] for sparse recovery of trigonometric polynomials on the circle. Since the (L_2 -normalized) spherical harmonic basis functions are not uniformly bounded, standard compressed sensing theory does not apply directly to sparse recovery on the sphere. Instead, we appeal to recent results in [6] concerning sparse recovery in orthonormal polynomial systems. Since orthonormal polynomials blow up sufficiently quickly and uniformly at the endpoints of their domain, one preconditions in order to transform the problem into that of sparse recovery in a uniformly bounded system. The decomposition of spherical harmonic basis functions into tensor products of trigonometric polynomials and Jacobi polynomials allows to prove the main result: any degree- D polynomial on the sphere (that is, with $N = D^2$ coefficients) consisting of at most s spherical harmonic basis elements can be efficiently recovered from $m \sim sN^{1/4} \log^4(N)$ independent sampling points drawn uniformly with respect to a certain measure (see below). We establish this result by verifying the restricted isometry property (RIP) of an associated random matrix.

2. BACKGROUND AND NOTATION

The spherical harmonics $Y_\ell^k, -\ell \leq k \leq \ell, \ell \geq 0$ form an orthonormal basis for the Hilbert space of square-integrable functions on the sphere. They are orthogonal with respect to the spherical surface measure Ω . In spherical coordinates $(\theta, \phi) \in B := [0, 2\pi) \times [0, \pi]$, ($x = \cos(\theta) \sin(\phi), y = \sin(\theta) \sin(\phi), z = \cos(\phi)$) $\in S^2$, this orthogonality relation becomes

$$\int_0^{2\pi} \int_0^\pi Y_\ell^k(\theta, \phi) \overline{Y_{\ell'}^{k'}(\theta, \phi)} \sin(\phi) d\phi d\theta = \delta_{\ell\ell'} \delta_{kk'}. \quad (1)$$

Here, \bar{z} denotes the complex conjugate and δ_{mn} is the Kronecker delta. The spherical harmonics may be expressed as

$$Y_\ell^k(\theta, \phi) = e^{ik\theta} (\sin \phi)^{|k|} p_{\ell-|k|}^{|k|}(\cos \phi), \quad (\theta, \phi) \in B, \quad (2)$$

where the Jacobi polynomials $(p_n^\alpha)_{n=0}^\infty$ with parameter $\alpha > -1$ are the orthonormal polynomial basis on the interval $[-1, 1]$ with respect to the measure $dv(x) = (1-x^2)^\alpha dx$ [9]. In particular, the Lebesgue measure $\alpha = 0$ generates the Legendre polynomials, while the Chebyshev measure, $\alpha = -1/2$, generates the Chebyshev polynomials.

Spherical harmonic expansions on the sphere are analogous to Fourier series expansions on the circle. Functions on the sphere of the form

$$g(\theta, \phi) = \sum_{\ell=0}^{D-1} \sum_{k=-\ell}^{\ell} c_{\ell,k} Y_\ell^k(\theta, \phi) \quad (3)$$

are called *harmonic polynomials* of degree $D - 1$. Note that $N = D^2$ spherical harmonic basis elements generate harmonic polynomials of degree $D - 1$. We will call a harmonic polynomial *s-sparse* if its coefficient vector $c = (c_{\ell,k}) \in \mathbb{C}^N$ has cardinality at most s ; i.e. $\|c\|_0 := |\{(\ell, k) : c_{\ell,k} \neq 0\}| \leq s$. More generally, the degree to which a harmonic polynomial can be well-approximated by its s most significant coefficients can

be quantified using the concept of best s -term approximation error which is defined, for a vector $z \in \mathbb{C}^N$ by

$$\sigma_s(z)_1 = \inf_{y: \|y\|_0 \leq s} \|y - z\|_1.$$

We say that a harmonic polynomial (3) is *compressible* if $\sigma_s(c)_1$ decays quickly as s increases.

3. MAIN RESULTS

We aim to recover sparse harmonic polynomials on the sphere from only a few function samples. Note that samples $y_j = g(\theta_j, \phi_j)$, $j = 1, \dots, m$, of a $D - 1$ -degree harmonic polynomial $g(\theta, \phi) = \sum_{\ell=0}^{D-1} \sum_{k=-\ell}^{\ell} c_{\ell,k} Y_{\ell}^k(\theta, \phi)$ may be expressed concisely in terms of the coefficient vector $c = (c_{\ell,k}) \in \mathbb{C}^N$ according to

$$y = \Phi c, \quad (4)$$

where Φ is the $m \times N$ matrix defined component-wise by

$$\Phi_{j,(k,\ell)} = Y_{\ell}^k(\theta_j, \phi_j). \quad (5)$$

We are interested in solving the system of linear equations (4) in the underdetermined setting $m < N$, and in particular, to single out the original sparse coefficient vector c from among the infinitely many solutions. The compressed sensing literature has suggested various reconstruction algorithms for sparse recovery; for simplicity we focus only on ℓ_1 -minimization [1] in this paper.

The L_{∞} -norm of the spherical harmonics $Y_{\ell}^k(\theta, \phi)$ increases with the degree ℓ according to $\sup_{-\ell \leq k \leq \ell} \|Y_{\ell}^k\|_{\infty} = \sup \|Y_{\ell}^0\|_{\infty} = \ell$, and this extremum is obtained at the spherical caps $\phi = 0, \pi$. This means that the linear system (4) is in general ill-conditioned. It is known (see Proposition 6) that $|(\sin \phi)^{1/2} Y_{\ell}^k(\theta, \phi)| \sim \ell^{1/4}$; consequently, we precondition the system (4) for numerical stability, multiplying both sides by the $m \times m$ diagonal matrix \mathcal{A} with entries $\mathcal{A}_{j,j} = (\sin \phi_j)^{1/2}$,

$$\mathcal{A}y = \mathcal{A}\Phi c. \quad (6)$$

Our main result is that any s -sparse harmonic polynomial on the sphere of maximal degree D can be recovered efficiently from a number of samples m that scales linearly with the sparsity level and sublinearly with the degree. This reconstruction is moreover robust with respect to noisy samples and passing from sparse to compressible vectors.

Theorem 1. *Let m, s , and N be given integers satisfying*

$$m \geq Cs \log^3(s) N^{1/4} \log(N). \quad (7)$$

Suppose that m coordinates on the sphere $(\theta_1, \phi_1), \dots, (\theta_m, \phi_m)$ are drawn independently from the uniform measure on $B = [0, 2\pi) \times [0, \pi]$.

Let Φ be the $m \times N$ spherical harmonic matrix (5) and let $\mathcal{A}\Phi$ be its preconditioned version (6). With probability exceeding $1 - N^{-\gamma \log^3(s)}$ the following holds for all harmonic polynomials $g(\theta, \phi) = \sum_{\ell=0}^{D-1} \sum_{k=-\ell}^{\ell} c_{\ell,k} Y_{\ell}^k(\theta, \phi)$. Suppose that noisy sample values $y_j = g(\theta_j, \phi_j) + \eta_j$ are observed, and that $\|\eta\|_{\infty} \leq \varepsilon$. Let

$$\hat{c} = \arg \min \|z\|_1 \quad \text{subject to} \quad \|\mathcal{A}\Phi z - \mathcal{A}y\|_2 \leq \sqrt{m}\varepsilon. \quad (8)$$

Then

$$\|c - \hat{c}\|_2 \leq \frac{C_1 \sigma_s(c)_1}{\sqrt{s}} + C_2 \varepsilon.$$

The constants C, C_1, C_2 , and γ are universal.

We expect that the bound (7) is not optimal, but so far we have not been able to remove the polynomial factor $N^{1/4}$.

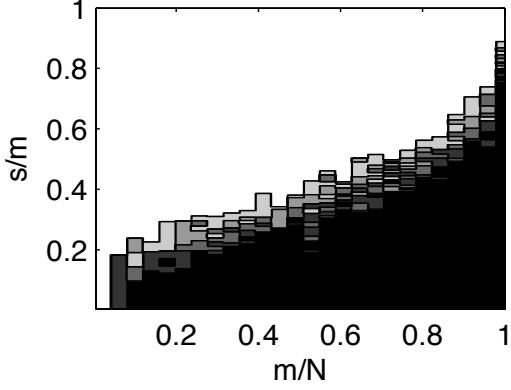
4. NUMERICAL EXPERIMENTS

In Figure 1, we plot a phase diagram illustrating the success of ℓ_1 -minimization (equation (8), $\varepsilon = 0$) in recovering s -sparse harmonic polynomials from m sample values. More precisely, we fix the degree to be $D = N^{1/2} = 16$ and vary the sparsity level s and number of measurements m . We form an s -sparse coefficient vector $c = (c_{\ell,k})$ by choosing a random support set from $\mathbb{N} \cap [1, N]$ of cardinality s and assigning independent and identically distributed Gaussian weights as the coefficients to this support. We draw m independent samples (θ_j, ϕ_j) from the uniform measure on B which we use to recover the s -sparse vector using ℓ_1 -minimization. For each pair (s, m) , we record the frequency of success of ℓ_1 -minimization out of 20 trials. The results, illustrated in Figure 1, show a sharp transition between uniform recovery (in black) and no recovery whatsoever (white). This transition curve is similar to the phase transition curves obtained for other compressive sensing matrices, e.g. the random partial discrete Fourier matrix or the Gaussian ensemble [2].

5. SPARSE RECOVERY VIA RESTRICTED ISOMETRY CONSTANTS

We prove Theorem 1 by showing that the preconditioned spherical harmonic matrix $\tilde{\Phi} = \mathcal{A}\Phi$ in (6) satisfies the *restricted isometry property* (RIP) [7, 1].

Fig. 1.



Definition 2 (Restricted isometry constants). Let $\Psi \in \mathbb{C}^{m \times N}$. For $s \leq N$, the restricted isometry constant δ_s associated to Ψ is the smallest number such that

$$(1 - \delta_s) \|c\|_2^2 \leq \|\Psi c\|_2^2 \leq (1 + \delta_s) \|c\|_2^2 \quad (9)$$

for all s -sparse vectors $c \in \mathbb{C}^N$.

Informally, we say that the matrix Ψ “has the restricted isometry property” if δ_s is small for s reasonably large compared to m . For matrices satisfying the restricted isometry property, the following ℓ_1 -recovery results can be shown [1, 3].

Theorem 3 (Sparse recovery for RIP-matrices). Let $\Psi \in \mathbb{C}^{m \times N}$. Assume that the restricted isometry constant of $\Psi \in \mathbb{C}^{m \times N}$ satisfies

$$\delta_{2s} < 3/(4 + \sqrt{6}) \approx 0.4652. \quad (10)$$

Let $x \in \mathbb{C}^N$ and assume noisy measurements $y = \Psi x + \eta$ are given with $\|\eta\|_2 \leq \varepsilon$. Let $x^\#$ be the minimizer of

$$\arg \min_{z \in \mathbb{C}^N} \|z\|_1 \text{ subject to } \|\Phi z - y\|_2 \leq \varepsilon. \quad (11)$$

Then

$$\|x - x^\#\|_2 \leq C_1 \frac{\sigma_s(x)_1}{\sqrt{s}} + C_2 \varepsilon, \quad (12)$$

for some constants $C_1, C_2 > 0$ that depend only on δ_{2s} . In particular, if x is s -sparse then reconstruction is exact, $x^\# = x$.

A general setup for matrices having the restricted isometry property are those associated to *bounded orthonormal systems* [7, 8, 1].

Theorem 4 (RIP for bounded orthonormal systems). Consider an orthonormal system of functions ψ_j , $j \in [1, N] \cap \mathbb{N}$ on a measurable space \mathcal{M} endowed with a probability measure ν , that is $\int_{\mathcal{M}} \psi_j \overline{\psi_k} d\nu = \delta_{j,k}$. Consider the matrix $\Psi \in \mathbb{C}^{m \times N}$ with entries

$$\Psi_{\ell,k} = \psi_k(x_\ell), \quad \ell \in [1, m] \cap \mathbb{N}, k \in [1, N] \cap \mathbb{N},$$

formed by i.i.d. samples x_ℓ drawn from the measure ν . Suppose this system has the uniform bound $K = \sup_{j \in [N]} \|\psi_j\|_\infty = \sup_{j \in [N]} \sup_{x \in \mathcal{D}} |\psi_j(x)|$. If

$$m \geq C \delta^{-2} K^2 s \log^3(s) \log(N), \quad (13)$$

then with probability at least $1 - N^{-\gamma \log^3(s)}$, the restricted isometry constant δ_s of $\frac{1}{\sqrt{m}} \Psi$ satisfies $\delta_s \leq \delta$. The constants $C > 0$ and $\gamma > 0$ are universal.

An important special case is the matrix associated to samples of the trigonometric system $(\exp(2\pi i j x))_{j=0}^{N-1}$ chosen from the uniform measure on $[0, 1]$, which has the optimal uniform bound $K = 1$. Another example is the sampling matrix associated to the Chebyshev polynomial system. In this case, $K = \sqrt{2}$.

6. SPARSE RECOVERY IN SPHERICAL HARMONIC SYSTEMS

Recall from (2) that the spherical harmonics can be expressed as tensor products of complex exponentials in θ and orthogonal polynomials in $\cos(\phi)$. Since the latter are not uniformly bounded, spherical harmonics do not fall directly into the scope of bounded orthonormal systems. To get around this obstacle, we proceed in a similar fashion to [6], and use estimates on the uniform rate of growth of orthogonal polynomials in order to precondition the spherical harmonic system.

First we will need the following growth estimates.

Proposition 5. Consider the weight function $v(x) = (1 - x^2)^\alpha$ on $[-1, 1]$, and let $(p_n^\alpha)_n$ be the associated orthonormal polynomial system. Then, for all $x \in [-1, 1]$, the following holds.

1. If $\alpha = 0$, the associated polynomials (p_n^0) are the Legendre polynomials and satisfy

$$(1 - x^2)^{1/4} |p_n^0(x)| \leq 2/\sqrt{\pi}. \quad (14)$$

2. For any $\alpha \geq 0$,

$$(1 - x^2)^{1/4 + \alpha/2} |p_n^\alpha(x)| \leq C_\alpha \quad (15)$$

3. If $\alpha \geq 3$, then

$$(1 - x^2)^{1/4 + \alpha/2} |p_n^\alpha(x)| \leq C \alpha^{1/6} \left(1 + \frac{\alpha}{n}\right)^{1/12}, \quad (16)$$

where $C > 0$ is a universal constant.

The bound (14) for Legendre polynomials is classical and known to be tight, and the more general bound (15) is also classical; see [9] for more details. The more refined bound (16) was derived only recently in [5]. Although the result is stated in [5] for the parameter range $\alpha \geq 1, n \geq 6$, one can verify that the key estimate (Lemma 8 in [4]) is valid also for the parameter range $\alpha \geq 3, n \geq 0$.

Using these bounds in conjunction with the tensor product representation (2), we arrive at the following rate of growth for the spherical harmonics.

Proposition 6. For all $\ell \in \mathbb{Z}^+$ and $-\ell \leq k \leq \ell$,

$$(\sin \phi)^{1/2} |Y_\ell^k(\theta, \phi)| \leq C(\ell + 1)^{1/4} \quad \forall x \in [-1, 1].$$

We can now state the proof of Theorem 1.

Proof of Theorem 1.

Consider, for $0 \leq \ell \leq D - 1, -\ell \leq k \leq \ell$, the functions

$$Q_\ell^k(\theta, \phi) = (\sin \phi)^{1/2} Y_\ell^k(\theta, \phi). \quad (17)$$

By Proposition 6,

$$\sup_{0 \leq \ell \leq \sqrt{N} - 1, -\ell \leq k \leq \ell} \|Q_\ell^k\|_\infty \leq CN^{1/8}$$

for a universal constant C . Because the spherical harmonics Y_ℓ^k are orthonormal with respect to the uniform measure $\sin(\phi)d\phi d\theta$, the Q_ℓ^k 's are orthonormal with respect to the product measure $d\theta d\phi$:

$$\begin{aligned} & \int_0^{2\pi} \int_0^\pi Q_\ell^k(\theta, \phi) Q_{\ell'}^{k'}(\theta, \phi) d\theta d\phi \\ &= \int_0^{2\pi} \int_0^\pi Y_\ell^k(\theta, \phi) Y_{\ell'}^{k'}(\theta, \phi) \sin(\phi) d\theta d\phi = \delta_{\ell\ell'} \delta_{kk'}. \end{aligned} \quad (18)$$

Applying Theorem 4 to the system $\{Q_\ell^k\}$, whose sampling matrix $Q_\ell^k(\theta_j, \phi_j) = y_j$ is equivalent to the preconditioned spherical harmonic matrix $(\sin \phi_j)^{1/2} Y_\ell^k(\theta_j, \phi_j) = y_j$ (6), Theorem 1 follows from the recovery results for restricted isometry systems in Theorem 3.

7. REFERENCES

- [1] E. J. Candes, J. Romberg, and T. Tao. Stable signal recovery from incomplete and inaccurate measurements. *Comm. Pure Appl. Math.*, 59(8):1207-1223, 2006.
- [2] D. Donoho and J. Tanner. Counting faces of randomly-projected polytopes when the projection radically lowers dimension. *Journal of the AMS*, 22(1):153, 2009.
- [3] S. Foucart. A note on guaranteed sparse recovery via ℓ_1 -minimization. *Appl. Comput. Harmon. Anal.*, 29(1):97103, 2010.
- [4] I. Krasikov, An upper bound on Jacobi polynomials, *Journal of Approximation Theory*, Vol. 149, Issue 2, 116-130, 2007.
- [5] I. Krasikov, On the Erdelyi-Magnus-Nevai conjecture for Jacobi polynomials, *Constructive Approximation*, vol. 28, no. 2, pp. 113-125, 2008.
- [6] H. Rauhut and R. Ward, Sparse Legendre expansions via ℓ_1 -minimization, submitted, 2010.
- [7] H. Rauhut, Compressive sensing and structured random matrices, in *Theoretical Foundations and Numerical Methods for Sparse Recovery*, Editor Massimo Fornasier, Radon Series Comp. Appl. Math, vol. 9, deGruyter, 2010.
- [8] M. Rudelson and R. Vershynin. On sparse reconstruction from Fourier and Gaussian measurements. *Comm. Pure Appl. Math*, 61:1025-1045, 2008.
- [9] G. Szegö. *Orthogonal Polynomials*. American Mathematical Society, Providence, RI, 1975.