Best time localized trigonometric polynomials and wavelets

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Abstract. In spaces of trigonometric polynomials, the minimum of the angular variance is determined, which is a time localization measure for \( L^2_{2\pi} \). Wavelets and wavelet packets are constructed with the resulting polynomials.

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1 Introduction

The classical Heisenberg uncertainty principle roughly states that a function on the real line cannot be arbitrarily localized both in the time and in the frequency domain. To have an analog, E. Breitenberger introduced the so-called angular variance and frequency variance for \( 2\pi \)-periodic functions (see [1]) which give in fact an uncertainty principle for \( L^2_{2\pi} \). Prestin, Quak, Rauhut, Selig [11] and independently Goh and Micchelli [5] established a connection to the classical Heisenberg uncertainty principle.

In this paper, we determine the minimum of the angular variance in the space \( T_n \) of all trigonometric polynomials of degree at most \( n \) and in the space \( T^m_n = T_n \odot T_{m-1} \). Properties of the resulting polynomials are investigated.

Similarly as in [6] and [13] where the uncertainty product of periodic wavelets and wavelet packets is investigated, we use, as an application, our optimal polynomials as periodic wavelets and wavelet packets. Surprisingly, we get a phenomenon which is somewhat related to the Balian-Low-Theorem (see [3, chapter 4]). Namely, a wavelet packet basis of \( L^2_{2\pi} \) of time optimal polynomials cannot be a Riesz basis and simultaneously have a uniformly bounded uncertainty product for all basis functions.

One can generalize the wavelet packet construction to obtain a basis which consists of functions in arbitrarily chosen frequency bands, whereas in the wavelet packet approach one does not have complete freedom in the choice of the frequency decomposition, i.e., the frequency bands. As basis functions one may again choose the time-optimal polynomials. The interested reader is referred to the thesis [12] where also the other results of this paper are taken from.

Let us now introduce the uncertainty measures we work with. We denote by \( L^2_{2\pi} \) the space of complex-valued square-integrable \( 2\pi \)-periodic functions with inner product \( \langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) g(t) dt \) and norm \( \| f \| = \sqrt{\langle f, f \rangle} = \sqrt{\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)|^2 dt} \).
Definition 1.1 For a function \( f \in L^2_{2\pi} \) represented by its Fourier series \( f = \sum_{s=-\infty}^{\infty} c_s e^{is\tau} \), we define the first trigonometric moment

\[
\tau(f) := \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\tau} |f(t)|^2 \, dt = \sum_{s=-\infty}^{\infty} c_s c_{s+1},
\]

and the angular variance

\[
\text{var}_A(f) := \frac{||f||^4 - |\tau(f)|^2}{||\tau(f)||^2} = \left| \sum_{s=-\infty}^{\infty} |c_s|^2 \right|^2 - 1.
\]

Furthermore, if \( f \) is absolutely continuous (\( f \in AC \) for short) and \( f' \in L^2_{2\pi} \), we define the frequency variance

\[
\text{var}_F(f) := \frac{||f'||^2}{||f||^2} - \frac{\text{var} \langle f', f \rangle}{||f||^4} = \frac{\sum_{s=-\infty}^{\infty} s^2 |c_s|^2}{\sum_{s=-\infty}^{\infty} |c_s|^2} - \left( \frac{\sum_{s=-\infty}^{\infty} s |c_s|^2}{\sum_{s=-\infty}^{\infty} |c_s|^2} \right)^2,
\]

and the uncertainty product

\[
U_{2\pi}(f) = \sqrt{\text{var}_A(f) \cdot \text{var}_F(f)}.
\]

We remark that the term \( \langle f', f \rangle \) in the frequency variance is always purely imaginary and hence vanishes for real-valued \( f \).

In [9] the uncertainty principle is shown, i.e., it holds \( U_{2\pi}(f) \geq \frac{1}{2} \) for every \( f \in AC \) with \( f' \in L^2_{2\pi} \), and, in addition, there exists no function for which equality is attained. But it is possible to construct a series of functions for which the uncertainty product tends to \( 1/2 \) which proves that the constant \( 1/2 \) is optimal (see [9] and [11]). An operator theoretical approach to uncertainty principles is given in [15] where the uncertainty principle on \( L^2_{2\pi} \) appears as a special case. In [12], this way of obtaining the uncertainty principle for \( L^2_{2\pi} \) (and also the classical Heisenberg one) is described in detail. Also [5] is devoted to a detailed study of the operator theoretical approach to uncertainty principles.

2 Best time localized polynomials in \( T_n \)

Trigonometric polynomials play an important role in applications because they are relatively easy to compute. Now, as we have time and frequency localization measures for \( 2\pi \)-periodic functions, one might ask for polynomials which are best localized, i.e., for which the variances are minimized. Concerning the frequency variance, the situation is completely clear, namely \( \text{var}_F(p) = 0 \) if and only if \( p \) is a monomial of the form \( ce^{is\tau} \). This and also the next section is devoted to the remaining problem of minimizing the angular variance in certain spaces of trigonometric polynomials.

We remark that one could also consider the uncertainty product, i.e., the problem of finding

\[
U_n := \min \{ U_{2\pi}(f) \mid f \in T_n \}.
\]

This problem leads to a very complicated expression that has to be minimized and, as far as known to the author, it is still unsolved.

Our aim is to find the polynomial \( \sigma_n \in T_n \) for which \( \text{var}_A(\sigma_n) \) is minimal and to compute

\[
M_n := \min \{ \text{var}_A(f) \mid f \in T_n \}.
\]

Since \( \text{var}_A(\alpha f) = \text{var}_A(f) \) for all \( \alpha \in \mathbb{C} \setminus \{0\} \) we only need to consider functions with norm 1. The minimum \( M_n \) exists since \( \{ f \in T_n \mid ||f|| = 1 \} \) is compact, \( \text{var}_A \) is continuous for \( \tau(f) \neq 0 \) and, furthermore, \( \text{var}_A(f) \to \infty \) as \( \tau(f) \to 0 \).
We remark that we use the symbol $\| \cdot \|_2$ for the Euclidean norm on $\mathbb{C}^n$ in the following.

Now, minimizing $\text{var}_A(f)$ for $f \in T_n$ is equivalent to maximizing

$$G(c) := \left| \sum_{s=-n}^{n-1} c_s e^{is+1} \right|$$

under the constraint $\| c \|_2^2 = \sum_{s=-n}^{n} | c_s |^2 = 1$ where $c = (c_s)_{s=-n}^{n-1} \in \mathbb{C}^{2n+1}$ denotes the vector of the Fourier coefficients of $f$. Writing $c_s$ in polar coordinates $c_s = r_s e^{i\phi_s}$, $r_s \geq 0, \phi_s \in \mathbb{R}$, we obtain

$$G(c) = \left| \sum_{s=-n}^{n-1} r_s e^{i\phi_s + is+1} \right|, \quad \| c \|_2^2 = \sum_{s=-n}^{n} r_s^2 = 1. \quad (2)$$

If $G(c)$ is maximal we necessarily need that $e^{i(\phi_s - \phi_{s+1})}$ is constant for all $s$. This is equivalent to the existence of an $\alpha \in [0, 2\pi)$ such that

$$(\phi_s - \phi_{s+1}) \mod 2\pi = \alpha, \quad s = -n, \ldots, n-1. \quad (3)$$

In this case

$$G(c) = \sum_{s=-n}^{n-1} r_s e^{is+1}. \quad (4)$$

Therefore, it is sufficient to consider $r_s \in \mathbb{R}_+$. From the solution of the simplified problem we get the general solution by choosing arbitrary $\phi_0, \alpha \in [0, 2\pi)$ and letting

$$\phi_s = \phi_0 - s\alpha, \quad s = -n, \ldots, n. \quad (5)$$

The simplified maximization problem is easily solved with the following lemma which is related to the theory of uniform approximation, in particular Jackson's theorems. Indeed, the first part of the proof follows an argument by Meinardus [8, p. 53].

**Lemma 2.1** For all $a = (a_k)_{k=0}^{n} \in \mathbb{R}^{n+1}$ it holds that

$$S_n(a) := \frac{\sum_{k=0}^{n-1} a_k a_{k+1}}{\sum_{k=0}^{n} a_k^2} \leq \cos \frac{\pi}{n+2}.$$  

Equality is attained for

$$a_k = \sin \frac{k+1}{n+2}, \quad k = 0, \ldots, n. \quad (6)$$

The optimal vector $a$ is unique up to multiplication by a constant.

**Proof:** Note that $S_n$ can be written in the form

$$S_n(x) = \frac{x^T A x}{x^T x}$$

where

$$A = \begin{pmatrix}
0 & 1/2 & 0 & \cdots & 0 \\
1/2 & 0 & 1/2 & \cdots & : \\
0 & \ddots & \ddots & \ddots & : \\
: & \ddots & 1/2 & 0 & 1/2 \\
0 & \cdots & 0 & 1/2 & 0
\end{pmatrix} \in \mathbb{R}^{(n+1) \times (n+1)}.$$
If \( \lambda_{\text{max}} \) is the largest eigenvalue of \( A \) it holds that

\[
\frac{x^T Ax}{x^T x} \leq \lambda_{\text{max}} \quad \text{for all} \quad x \in \mathbb{R}^{n+1}.
\]

Equality is attained if we choose an eigenvector \( x_{\text{max}} \) that belongs to \( \lambda_{\text{max}} \).

The eigenvalue equation for an eigenvalue \( \lambda \) and an eigenvector \( a \in \mathbb{R}^{n+1} \) leads to a three term recursion for \( \lambda \). It is a fairly standard result then that this three term recursion is solved by the \( n+1 \) zeros of the Chebychev polynomials of the second kind \( U_{n+1} \) and that these zeros are the only solutions. For more details see [12] or [8, p. 53]. Now, for \( 0 \leq \theta \leq \pi \) the Chebychev polynomial is given by

\[
U_{n+1}(\cos \theta) = \frac{\sin((n+2)\theta)}{\sin \theta}\]

whose largest zero clearly is

\[
\lambda_{\text{max}} = \cos \frac{\pi}{n+2}.
\]

Since \( A \) has \( n+1 \) distinct eigenvalues every eigenspace has dimension 1, which yields the last assertion of the lemma.

What remains to show is the optimality of (6). Hence, for this choice of \( a \), we compute

\[
\sum_{k=0}^{n-1} a_k a_{k+1} = \sum_{k=0}^{n} \frac{(k+2)\pi}{n+2} \sin \left( \frac{(k+1)\pi}{n+2} \right) = \sum_{k=0}^{n} \frac{k\pi}{n+2} \sin \left( \frac{(k+1)\pi}{n+2} \right) \tag{7}
\]

\[
= \frac{1}{2} \sum_{k=0}^{n} \left( \sin \left( \frac{(k+2)\pi}{n+2} \right) + \sin \left( \frac{k\pi}{n+2} \right) \right) \sin \left( \frac{(k+1)\pi}{n+2} \right)
\]

\[
= \cos \frac{\pi}{n+2} \sum_{k=0}^{n} \sin^2 \left( \frac{(k+1)\pi}{n+2} \right) = \cos \frac{\pi}{n+2} \sum_{k=0}^{n} a_k^2.
\]

Thus,

\[
S_n(a) = \frac{\sum_{k=0}^{n-1} a_k a_{k+1}}{\sum_{k=0}^{n} a_k^2} = \cos \frac{\pi}{n+2}.
\]

Now we are able to prove the following characterization.

**Theorem 2.2** It holds that

\[
M_n = \min \{ \text{var}_A(f) \mid f \in T_n \} = \tan^2 \frac{\pi}{2n+2}.
\]

Furthermore, the minimum is attained for

\[
\sigma_n(t) := 1 + 2 \sum_{s=1}^{n} \cos \frac{s\pi}{2n+2} \cos st. \tag{8}
\]

The minimal polynomial is unique up to translation and multiplication by a scalar \( c \in \mathbb{C} \setminus \{0\} \).

**Proof:** By reindexing we get from the preceding lemma

\[
\sum_{s=-n}^{n-1} c_s c_{s+1} \leq \cos \frac{\pi}{2n+2}.
\]

Hence, for \( f \in T_n \) with positive Fourier coefficients \( c_s = c_s(f) \) we have

\[
\text{var}_A(f) = \left( \frac{\sum_{s=-n}^{n-1} c_s^2}{\sum_{s=-n}^{n-1} c_s c_{s+1}} \right)^2 \geq \frac{1}{\cos^2 \frac{\pi}{2n+2}} - 1 = \tan^2 \frac{\pi}{2n+2}. \tag{9}
\]

By permitting arbitrary complex-valued Fourier coefficients we will not decrease \( \text{var}_A(f) \) as follows from (1), (2) and (4).
Figure 1: Time optimal trigonometric polynomials.

By Lemma 2.1, equality is attained in (9) iff

\[ c_s = c_s(\sigma_n) := \begin{cases} \cos \frac{s\pi}{2n+2}, & s = -n, \ldots, n, \\ 0, & \text{otherwise}, \end{cases} \]  

(10)

where the reindexing turns the sine-terms in (6) into the cosine-terms in (10).

By (3) and (5) we get all solutions of the minimization problem by fixing \( \phi_0, \alpha \in \mathbb{R} \), \( b \in \mathbb{C} \setminus \{0\} \) and letting

\[ c_s(\sigma_n) := b c_s(\sigma_n) e^{i\phi_0 + i\alpha s} = \frac{b e^{i\phi_0}}{c} c_s(\sigma_n) e^{i\alpha s}, \quad s = -n, \ldots, n. \]

This is equivalent to \( \sigma_n^*(t) := c \sigma_n(\tau + \alpha) \).

Using the identity

\[ D_n(t) := 1 + 2 \sum_{k=1}^{n} \cos kt = \frac{\sin(2n+1)t/2}{\sin t/2} \]  

(11)

for the Dirichlet-kernel \( D_n \) (and its twice differentiated form) we can compute the norm, the frequency variance and the uncertainty product of \( \sigma_n \):

\[ ||\sigma_n||^2 = n + 1, \]

\[ \text{var}_F(\sigma_n) = ||\sigma_n||^{-2} \sum_{s=-n}^{n} s^2 |\sigma_n|^2 = \frac{n(n+2)}{3} - \frac{1}{2} \cos^2 \frac{\pi}{2n+2}, \]

\[ U_{2\pi}^2(\sigma_n) = \text{var}_A(\sigma_n) \text{var}_F(\sigma_n) = \frac{n(n+2)}{3} \tan^2 \frac{\pi}{2n+2} - \frac{1}{2}, \]

In the limit we get

\[ \lim_{n \to \infty} U_{2\pi}(\sigma_n) = \sqrt{\frac{\pi^2}{12} - \frac{1}{2}} \approx 0.5679. \]  

(12)

We remark that another way to obtain this limit without computing \( U_{2\pi}(\sigma_n) \) explicitly is to make use of the results in [11]. There we have shown that for a series \( u_n \) of \( 2\pi \)-periodic
functions given by

\[ u_n(t) := \sum_{k=-\infty}^{\infty} h \left( \frac{k}{n} \right) e^{ikt} \]  

(13)

where \( h \in C_0 \cap L^2(\mathbb{R}) \) with some additional smoothness conditions, it holds that

\[ \lim_{n \to \infty} U_{2n}(u_n) = U_{R}(h) \]

where \( U_{R} \) is the uncertainty product on \( L^2(\mathbb{R}) \), defined by

\[ U_{R}(f) := \left( \| f \|^2 - \frac{\| \mathcal{K}f \|^2}{\| f \|^2} \right) \left( \| f' \|^2 - \frac{\| \mathcal{K}f' \|^2}{\| f' \|^2} \right) \| f \|^{-4}. \]

Hereby, the norm and the scalar product denote the usual ones in \( L^2(\mathbb{R}) \). We further remark that the above result (in slightly different form) was obtained independently in [5, Theorem 3.6].

Observe now that by setting

\[ h(t) := \begin{cases} \cos \frac{\pi t}{2}, & t \in [-1, 1], \\ 0, & t \notin [-1, 1] \end{cases} \]  

(14)

we obtain our functions \( \sigma_n \). More precisely,

\[ \sigma_{n-1}(t) = \sum_{k=-n}^{n} h \left( \frac{k}{n} \right) e^{ikt}. \]

Theorem 6.1 in [11] then yields

\[ \lim_{n \to \infty} U_{2n}(\sigma_n) = U_{R}(h) = \sqrt{\frac{\pi^2}{12} - \frac{1}{2}}. \]

In fact, this way to obtain the limit is easier since the computation of \( U_{R}(h) \) involves only three simple integrals.

The computation of the following two other representations of \( \sigma_n \) is straightforward, see also [12].

**Lemma 2.3** The functions \( \sigma_n \) can be expressed by

\[ \sigma_n(t) = \frac{1}{2} \left( D_n \left( t + \frac{2\pi}{2n+2} \right) + D_n \left( t - \frac{2\pi}{2n+2} \right) \right) \]

\[ = \begin{cases} \frac{\sin \frac{\pi}{2n+2} \cos \left( \frac{n+1}{2} t \right)}{n+1} & \text{for } t \neq \pm \frac{2\pi}{2n+2} + 2\pi k, \; k \in \mathbb{Z}, \\ \frac{\sin \frac{\pi}{2n+2} \cos \left( \frac{n+1}{2} t \right)}{n+1} & \text{for } t = \pm \frac{2\pi}{2n+2} + 2\pi k, \; k \in \mathbb{Z} \end{cases} \]  

(15)

where \( D_n \) denotes the Dirichlet kernel.

The last lemma also shows a connection to the Feket-Korovkin kernel defined by

\[ K_n(t) := \frac{2}{n+2} \left( \frac{\sin \frac{\pi}{2n+2} \cos \left( \frac{n+1}{2} t \right)}{\cos t - \cos \frac{\pi}{n+2}} \right)^2. \]

Observe that \( \sigma_n(t) = (n+1)K_{2n}(t) \).
We end this section with the discussion of some properties of \( \sigma_n \). By using the representation (15) it is not difficult to verify that
\[
\lim_{n \to \infty} \sigma_n(t) = 0 \quad \text{for} \ t \notin 2\pi \mathbb{Z},
\]
\[
\lim_{n \to \infty} \sigma_n(2\pi k) = \lim_{n \to \infty} \cot \frac{\pi}{4n+4} = \infty.
\]
Furthermore, by making use of some results in \([4]\), in particular Lemmas 1 and 3, it is easily shown that
\[
||\sigma_n||_{L^2_{2\pi}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\sigma_n(x)| dx \leq \frac{4\sqrt{2}}{\pi}.
\]
The uniform boundedness of the \( L^1_{2\pi} \)-norm of \( \sigma_n \) gives reason to investigate the properties as convolution kernel. And in fact, it holds that \( \sigma_n * f \to f \) for all \( f \in C_{2\pi}, L^p_{2\pi}, \ 1 \leq p < \infty \). This and some more results concerning \( \sigma_n \) as convolution kernel can be easily derived by observing the close similarity to the Rogosinski-kernel which is defined by
\[
R_n(t) := \frac{1}{2} \left( D_n \left( t + \frac{2\pi}{2n+1} \right) + D_n \left( t - \frac{2\pi}{2n+1} \right) \right) = 1 + 2 \sum_{s=1}^{n} \cos \frac{s\pi}{2n+1} \cos st.
\]
In particular Theorem 2.4.8 and Theorem 2.4.9 in \([2]\) also hold for \( \sigma_n \) instead of \( R_n \). The proofs work out analogously. For details see also \([12]\).

3 Best time-localized polynomials in \( T^m_m \)

For the construction of trigonometric wavelets and wavelet packets the space
\[
T^m_m = T_n \cap T_{m-1} = \{ f \in L^2_{2\pi} \mid c_k(f) = 0, \ |k| \neq m, \ldots, n \}, \quad \text{for} \ n \geq m \geq 1.
\]
becomes of great importance. So now, we transfer our problem to this space, i.e., we want to find the best time localized trigonometric polynomials in \( T^m_m \) and to calculate
\[
M^m_m := \min \{ \operatorname{var}_A(f) \mid f \in T^m_m \}.
\]
Similarly as in Section 2, we conclude that this problem is equivalent to maximizing
\[
G(c) := \left| \sum_{s=-n}^{m-1} c_sc_{s+1} + \sum_{s=m}^{n-1} c_sc_{s+1} \right| = \left| \sum_{s=m}^{n-1} (c_sc_{s+1} + c_{s+1}) \right|
\]
under the constraint
\[
||c||_2^2 = \sum_{s=m}^{n} (|c_s|^2 + |c_{s+1}|^2) = 1
\]
where \( c = (c_{-n}, c_{-n+1}, \ldots, c_m, c_{m+1}, \ldots, c_n) \). By passing over to polar coordinates \( c_s = r_s e^{i\theta_s} \) where \( r_s \in \mathbb{R}_+, \ \phi_s \in \mathbb{R} \) we obtain
\[
G(c) = \left| \sum_{s=m}^{n} \left( r_s e^{i\phi_s} - (s+1) e^{i\phi_{s+1}} - \phi_s \right) + r_s e^{i\phi_{s+1}} e^{i\phi_{s+1}} \right|,
\]
\[
||c||_2^2 = \sum_{s=m}^{n} (r_s^2 + r_{s+1}^2) = 1.
\]

Analogously as in Section 2 we conclude that if \( c \) maximizes \( G \), then there must necessarily exist an \( \alpha \in [0, 2\pi) \), such that
\[
(\phi_s - \phi_{s+1}) \mod 2\pi = \alpha, \quad s = m, \ldots, n-1,
\]
\[
(\phi_{(s+1)} - \phi_{(s)}) \mod 2\pi = \alpha, \quad s = m, \ldots, n-1.
\]
For the same reason as in Section 2 we only consider positive values of \( r_s \) at first. Hence, we maximize

\[
G(r) = \sum_{s=m}^{n-1} (r_{s}\overrightarrow{r}_{s+1}) + r_s r_{s+1}) = F^{n-m+1}(r_-) + F^{n-m+1}(r_+)
\]

where

\[
r_- := (r_{m-1}, r_{m-2}, \ldots, r_{1}) \in \mathbb{R}^{n-m+1}_+,
\]

\[
r_+ := (r_{m}, r_{m+1}, \ldots, r_{n}) \in \mathbb{R}^{n-m+1}_+,
\]

and

\[
F^n(x) := \sum_{s=1}^{n-1} x_s x_{s+1}, \quad x \in \mathbb{R}^n.
\]

The following statement whose proof is straightforward helps us to apply the results of Section 2 to our new problem. Note that all norms denote the Euclidean norm on \( \mathbb{C}^n \) in the next lemma.

**Lemma 3.1** Let \( H : \mathbb{C}^n \to \mathbb{R}^+ \) be a continuous function such that \( H(\alpha x) = |\alpha|^2 H(x) \) for \( \alpha \in \mathbb{C} \). Then

\[
\max_{||x||=1} H(x) = \max_{||x||=1} H(x)
\]

and furthermore, if \( x_i \), for \( i \in I \), are the maxima of \( H(x) \) under the constraint \( ||x|| = 1 \) then \( (\alpha x_i, \overline{\alpha x}_i) \), for \( i \in I \), \( \alpha, \beta \in \mathbb{C} \), \( |\alpha|^2 + |\beta|^2 = 1 \), are the maxima of \( H^*(x,y) := H(x) + H(y) \) under the constraint \( ||x||^2 + ||y||^2 = 1 \) and there do not exist any other maxima of \( H^* \).

Note that \( F^n \) is indeed continuous and satisfies the condition \( F(\alpha x) = \alpha^2 F(x) \) for \( \alpha \in \mathbb{R}^+ \) (here we do not need complex values of \( \alpha \)). Therefore, we can apply the results of Section 2 to get the following theorem.

**Theorem 3.2** It holds that

\[
M^n_m = \min \{ \text{var}_A(f) \mid f \in \mathbb{C}^n \} = \tan^2 \frac{\pi}{n-m+2}. \tag{17}
\]

Moreover, all functions for which the minimum is attained are given by

\[
\zeta^n_m(t) = d_1 \sum_{s=m}^{n} \sin \left( \frac{s-m+1}{n-m+2} \pi e^{i(s-\beta)} \right) + d_2 \sum_{s=m}^{n} \sin \left( \frac{s-m+1}{n-m+2} \pi e^{-i(s-\beta)} \right) \tag{18}
\]

where \( d_1, d_2 \in \mathbb{C} \), \( (d_1, d_2) \neq (0,0) \), \( \beta \in \mathbb{R} \).

**Remark 3.3** All real-valued functions \( \zeta^n_m \) are given by

\[
\zeta^n_m(t) = a \sum_{s=m}^{n} \sin \left( \frac{s-m+1}{n-m+2} \pi \cos(s(t-\beta) + \gamma) \right)
\]

where \( a \in \mathbb{R} \setminus \{0\} \), \( \beta, \gamma \in \mathbb{R} \). Up to multiplication with a constant the only even polynomial among these is

\[
\sigma^n_m(t) := 2 \sum_{s=m}^{n} \sin \left( \frac{s-m+1}{n-m+2} \pi \cos st \right).
\]
PROOF OF THEOREM 3.2: According to Lemma 2.1 and Lemma 3.1 we choose

\[ r_s = a_1 \sin \left( \frac{(s-m+1)n}{n-m+1} \pi \right), \quad s = m, \ldots, n, \]

\[ r_{-s} = a_2 \sin \left( \frac{(s-m+1)n}{n-m+1} \pi \right), \quad s = m, \ldots, n \]

with \((a_1, a_2) \in \mathbb{R}^2 \setminus (0, 0)\). Furthermore, let

\[ c_s = c_s(c_m^n) = r_s e^{i(\gamma_1 - s\beta)}, \quad a_1 e^{i\gamma_1} \sin \left( \frac{(s-m+1)n}{n-m+1} \pi \right) e^{-i\beta}, \quad s = m, \ldots, n, \]

\[ c_{-s} = c_{-s}(c_m^n) = r_{-s} e^{i(\gamma_2 + s\beta)} = a_2 e^{i\gamma_2} \sin \left( \frac{(s-m+1)n}{n-m+1} \pi \right) e^{i\beta}, \quad s = m, \ldots, n \]

where \(\gamma_1, \gamma_2, \beta \in \mathbb{R}\). Then, condition (16) is satisfied and these are all possible solutions. Setting \(d_i = a_i e^{i\gamma_i} (i = 1, 2)\) yields (18). The first assertion of the lemma follows easily with the first part of Lemma 2.1.

Similarly as for \(\sigma_n\) we compute the norm, the frequency variance and the uncertainty product of \(\sigma_m^n\) yielding

\[ \|\sigma_m^n\|_{L_2} = n - m + 2, \]

\[ \text{var}_{F}(\sigma_m^n) = \frac{(n-m)(n-m+1)}{3} + mn \frac{1}{2} \cot^2 \frac{\pi}{n-m+2}, \]

\[ U_{2\pi}(\sigma_m^n) = \left( \frac{(n-m)(n-m+1)}{3} + mn \right) \tan^2 \frac{\pi}{n-m+2} - \frac{1}{2}. \tag{19} \]

For the special case \(n = 2m\) we get the limit

\[ \lim_{m \to \infty} U_{2\pi}(\sigma_m^{2m}) = \sqrt{\frac{7\pi^2}{3} - \frac{1}{2}} \approx 4.7465. \]

4 Construction of time optimal wavelets

One main scope of wavelet theory is to provide bases for certain function spaces, e.g., \(L^2(\mathbb{R})\) or \(L^2_{2\pi}\), which have the property that all basis functions are well localized both in the time
domain and frequency domain. Of course, by the uncertainty principle they cannot be arbitrarily localized in both domains.

Now, by construction, the functions $\sigma_j$ and $\sigma^{\text{ng}}_j$ from Sections 2 and 3 are well localized in the frequency domain, since only a finite number of Fourier coefficients are non-zero, and, of course, they are well localized in the time domain by construction. The obvious question arises whether these functions can be used as periodic wavelets. In fact, it is possible to construct ‘time optimal wavelets’, as we will see.

We start by briefly introducing periodic wavelets. For a detailed discussion we refer to [7], [10] and [14].

For arbitrary $c \in \mathbb{N}$ and $j \in \mathbb{N}_0$ let

$$ N_j := c 2^j. $$

Note that in [7] the number $c$ is omitted, i.e., $c = 1$.

The following definition plays a key role in wavelet theory.

**Definition 4.1** A multiresolution analysis (MRA) of $L^2_{2\pi}$ is a sequence of subspaces $\{V_j\}_{j \in \mathbb{N}_0}$ of $L^2_{2\pi}$ with the following properties.

(MR1) For all $j \in \mathbb{N}_0$ there exist a function $\phi_j \in V_j$, such that

$$ \{\phi_j \left( \cdot - \frac{2^j \pi}{N_j} \right) : s = 0, \ldots, 2N_j - 1 \} $$

is a basis for $V_j$.

(MR2) $V_j \subset V_{j+1}$ for all $j \in \mathbb{N}_0$.

(MR3) $\text{clos}_{L^2_{2\pi}} (\bigcup_{j \in \mathbb{N}_0} V_j) = L^2_{2\pi}$.

The function $\phi_j$ is called a scaling function and $V_j$ is called a sampling space. We say that $\{\phi_j : j \in \mathbb{N}_0\}$ generates a multiresolution analysis if $V_j = \text{span}\{\phi_j \left( \cdot - \frac{2^j \pi}{N_j} \right) : s = 0, \ldots, 2N_j - 1\}$ satisfies (MR1) – (MR3). In [14, Chapter 3] and also in [7] we find criteria on the Fourier coefficients of $\phi_j$ that make sure that (MR1) – (MR3) hold. These are used in the proof of Lemmas 4.6 and 4.7.

In view of (MR1) we define the translation operator $T_j : L^2_{2\pi} \to L^2_{2\pi}$ by

$$ T_j f(x) := f \left( x - \frac{\pi}{N_j} \right) \quad \text{for } j \in \mathbb{N}_0. $$

We say that a subspace $X \subset L^2_{2\pi}$ is $T_j$-invariant if for all $f \in X$ it holds that $T_j f \in X$.

**Definition 4.2** Given a multiresolution analysis $\{V_j\}_{j \in \mathbb{N}_0}$ of $L^2_{2\pi}$, the wavelet space $W_j$, for $j \in \mathbb{N}_0$, is defined as the orthogonal complement of $V_j$ in $V_{j+1}$, i.e., $W_j := V_{j+1} \ominus V_j$.

One can show that every wavelet space $W_j$ is $T_j$-invariant, and that there exist a function $\psi_j \in W_j$ such that the translates $\{T_j^{\ell} \psi_j : s = 0, \ldots, 2N_j - 1\}$ form a basis of $W_j$ (see [14, Corollary 3.1.6]). Such a function $\psi_j$ is called a wavelet.

It makes sense to decompose the wavelet spaces, too, which yields wavelet packet spaces and wavelet packets. In [14] the following definition can be found.

**Definition 4.3** Given a wavelet space $W_j = W_j^{0,0}$, for $j \in \mathbb{N}$ the wavelet packet spaces $W_j^{\lambda,p}$, $p = 0, \ldots, 2^\lambda - 1$ of level $j$ and depth $\lambda$, $0 \leq \lambda \leq j$ are subspaces of $L^2_{2\pi}$ which have the following properties.

(WP1) For all $p = 0, \ldots, 2^\lambda - 1$ there exists a function $\psi_j^{\lambda,p} \in W_j^{\lambda,p}$ such that

$$ \{T_j^{\ell} \psi_j^{\lambda,p} : s = 0, \ldots, 2N_j - \lambda - 1\} $$

is a basis for $W_j^{\lambda,p}$.

(WP2) For all $\ell = 0, \ldots, \lambda - 1$ and $p = 0, \ldots, 2^\ell - 1$ it holds that

$$ W_j^{\ell,p} = W_j^{\ell+1,2p} \oplus W_j^{\ell+1,2p+1}. $$
The functions $\psi_j^{\lambda p}$ are called wavelet packet functions or, simply, wavelet packets.

With this setting we obtain an iterative decomposition of the wavelet space $W_j^{0,0}$ into subspaces $W_j^{\lambda p}$,

$$W_j = \bigoplus_{p=0}^{2^j-1} W_j^{\lambda p}$$

for all possible $\lambda$.

For notational convenience we introduce multi-indices $\alpha = (j, \lambda, p)$ and define $W^\alpha := W_j^{\lambda p}$. In order to avoid distinguishing between sampling spaces and wavelet packet spaces we additionally set $W_j^{0,-1} := V_j$ and furthermore, $N^\alpha := N_{j,\lambda}$, $T_\alpha := T_{j,\lambda}$, for $\alpha = (j, \lambda, p)$, $\psi^\alpha = \psi_j^{\lambda p}$ for $\alpha \neq (j,0,-1)$, and $\psi^\alpha = \psi_j^{0,-1} = \phi_j$ for $\alpha = (j,0,-1)$.

With the introduction of wavelet packet spaces, a variety of ways to decompose $L^2_{2\pi}$ into a direct sum of subspaces comes up. Therefore, it is useful to make the following definition.

**Definition 4.4** A collection

$$D = \{V_{j_0} = W^{\alpha_0}, \ W^{\alpha_1}, \ W^{\alpha_2}, \ldots \}$$

(clearly, $\alpha_0 = (j_0,0,-1)$) of one sampling space, wavelet spaces and wavelet packet spaces is said to be a wavelet packet decomposition of $L^2_{2\pi}$ if

$$W^\alpha \cap W^\beta = \{0\} \quad \text{for} \quad \alpha, \beta \in I_D, \alpha \neq \beta,$$

$$\text{clos}_{L^2_{2\pi}} \bigoplus_{\alpha \in I_D} W^\alpha = L^2_{2\pi}$$

where

$$I_D := \{\alpha \ | \ W^\alpha \in D\}.$$  

When we recall the tree model we can think of a wavelet packet decomposition as the collection of all "leaves" of the decomposition tree.

A wavelet packet decomposition $D$ generates a Schauder basis $B$ of $L^2_{2\pi}$ in a natural way,

$$B := \bigcup_{\alpha \in I_D} \{T_\alpha^s \psi^\alpha \ | \ s = 0, \ldots, 2^{N^\alpha} - 1\}.$$  \hspace{1cm} (21)

Each $f \in L^2_{2\pi}$ can be uniquely written as

$$f = \sum_{i=0}^{\infty} \sum_{s=0}^{2^{N^{\alpha_i}}-1} c_{i,s} T_{\alpha_i}^s \psi^{\alpha_i}$$

where $I_D = \{\alpha_i \ | \ i \in \mathbb{N}\}$. Note that convergence is in $L^2_{2\pi}$ and depends on the order of summation, in general. However, in case of a Riesz basis the convergence becomes independent of the order of summation. Recall that a series $(f_n)_{n \in \mathbb{N}}$ in $L^2_{2\pi}$ is called a Riesz basis of $L^2_{2\pi}$ if $(f_n)_{n \in \mathbb{N}}$ is complete, i.e. \text{clos}_{L^2_{2\pi}} \text{span}\{f_n\} = L^2_{2\pi}$ and if there exist numbers $0 < C, D < \infty$ - called Riesz constants - such that for every sequence $(a_n)_{n \in \mathbb{N}} \in \ell^2(\mathbb{N})$ it holds that

$$C \sum_{n \in \mathbb{N}} |a_n|^2 \leq \left\| \sum_{n \in \mathbb{N}} a_nf_n \right\|_{L^2_{2\pi}}^2 \leq D \sum_{n \in \mathbb{N}} |a_n|^2.$$
Now, we define
\[
C^\alpha = 2N^\alpha \min_{k=0, \ldots, 2N^\alpha-1} \sum_{m \in \mathbb{Z}} |c_{k+2N^\alpha m}(\psi^\alpha)|^2, \quad (22)
\]
\[
D^\alpha = 2N^\alpha \max_{k=0, \ldots, 2N^\alpha-1} \sum_{m \in \mathbb{Z}} |c_{k+2N^\alpha m}(\psi^\alpha)|^2. \quad (23)
\]

Given a wavelet packet decomposition $D$ of $L^2_{2\pi}$, let
\[
C := \inf_{\alpha \in D} C^\alpha, \quad D := \sup_{\alpha \in D} D^\alpha.
\]

It is possible to prove that $C, D$ are the optimal Riesz constants for the basis $B$, which is generated by the wavelet packet decomposition $D$ (see also [14, Lemma 2.1.4(0)]). So if $0 < C, D < \infty$ then $D$ generates a Riesz basis.

Before we turn over to the construction of 'time optimal wavelets and wavelet packets' we provide results which can be applied to a slightly more general class of polynomials than our time optimal ones. Therefore, let us first make the following definition.

**Definition 4.5** The frequency support of a function $f \in L^2_{2\pi}$ is the set of integer numbers defined by
\[
\text{supp} \hat{f} := \{ k \mid c_k(f) \neq 0 \}.
\]

For simpler notation we write $\text{supp} \hat{f} = [a, b]$, for $a, b \in \mathbb{Z}$, if $\text{supp} \hat{f} = \{a, a+1, a+2, \ldots, b\}$, and $\text{supp} \hat{f} = [\pm a, b]$, for $a, b \in \mathbb{N}$, if $\text{supp} \hat{f} = \{\pm a, \pm(a+1), \pm(a+2), \ldots, \pm b\}$.

**Lemma 4.6** Let $\phi_j, \psi_j \in L^2_{2\pi}, \ j \in \mathbb{N}_0$, be trigonometric polynomials.

a) If
\[
\text{supp} \hat{\phi}_j = [-N_j, N_j]
\]
then $\{\phi_j : j \in \mathbb{N}_0\}$ generates a multiresolution analysis.

b) If furthermore
\[
\text{supp} \hat{\psi}_j = \pm [N_j, N_{j+1}]
\]
and
\[
\overline{c_{N_j}(\hat{\phi}_j)} \ c_{N_j}(\psi_j) = -c_{-N_j}(\hat{\phi}_j) \ c_{-N_j}(\psi_j)
\]  
(24)

then the functions $\psi_j$ are wavelets corresponding to the MRA in a).

**Proof:** The proof is straightforward using Corollaries 3.1.2, 3.1.3 and Theorem 3.1.4 in [14] or the corresponding results in [7]. For the details see also [12]. \[Q.E.D.\]

Before we state a similar lemma for the wavelet packets we introduce the notation
\[
N_j^{\lambda \psi} := N_j + p N_{-j-\lambda}.
\]
Lemma 4.7 Let \( \psi_j^{\lambda p} \in L^2_{2\pi} \), for \( j \in \mathbb{N}_0 \), be trigonometric polynomials with

\[
\text{supp} \hat{\psi}_j^{\lambda p} = \pm [N_j^{\lambda p}, N_j^{\lambda p+1}].
\]

If \( \psi_j^{0,0} \) are wavelets corresponding to some MRA and if, furthermore, for even \( p, 1 \leq \lambda \leq j \), it holds that

\[
c_M(\psi_j^{\lambda p}) c_M(\psi_j^{\lambda p+1}) = -c_{-M}(\psi_j^{\lambda p}) c_{-M}(\psi_j^{\lambda p+1})
\]  

(25)

and

\[
\frac{c_N(\psi_j^{\lambda p})}{c_N(\psi_j^{\lambda -1p/2})} = \frac{c_{-N}(\psi_j^{\lambda p})}{c_{-N}(\psi_j^{\lambda -1p/2})}, \quad (26)
\]

\[
\frac{c_K(\psi_j^{\lambda p+1})}{c_K(\psi_j^{\lambda -1p/2})} = \frac{c_{-K}(\psi_j^{\lambda p+1})}{c_{-K}(\psi_j^{\lambda -1p/2})}, \quad (27)
\]

where \( N = N_j^{\lambda p} \), \( M = N_j^{\lambda p+1} \), \( K = N_j^{\lambda p+2} \), then \( \psi_j^{\lambda p} \) are wavelet packets.

**Proof:** The proof is straightforward using Corollaries 3.2.2 and 3.2.3 in [14]. For details we again refer to [12].

---

The next result is helpful for the construction of some special polynomial wavelets and wavelet packets.

**Theorem 4.8** Let \( j \in \mathbb{N}_0 \), \( 0 \leq \lambda \leq j \), \( 0 \leq p \leq 2^\lambda - 1 \). If \( \phi_j, \eta_j^{\lambda p} \in L^2_{2\pi} \) are real-valued even trigonometric polynomials with

\[
\text{supp} \phi_j = [-N_j, N_j], \quad \text{supp} \eta_j^{\lambda p} = \pm [N_j^{\lambda p}, N_j^{\lambda p+1}]
\]

then \( \{ \phi_j : j \in \mathbb{N}_0 \} \) generates a multiresolution analysis and the functions

\[
\psi_j = \psi_j^{0,0} := T_{j+1} \eta_j^{0,0} = \eta_j^{0,0} \left( \cdot - \frac{\pi}{N_{j+1}} \right)
\]

are wavelets corresponding to this MRA. Furthermore, the functions

\[
\psi_j^{\lambda p} := T_{j+1}^{p_0} T_{j+2}^{p_1} T_{j+3}^{p_2} \cdots T_{j+p_0-1}^{p_0} T_{j+1} \eta_j^{\lambda p}
\]  

(28)

are wavelet packets.

**Proof:** Since \( \phi_j, \eta_j^{\lambda p} \) are real-valued and even it holds for all \( k \in \mathbb{Z} \) that

\[
c_k(\phi_j) = c_{-k}(\phi_j) \in \mathbb{R}, \quad c_k(\eta_j^{\lambda p}) = c_{-k}(\eta_j^{\lambda p}) \in \mathbb{R}.
\]

Furthermore, by setting \( \eta_j^{0,0} = \eta_j^{0,0} \) we have

\[
c_{N_{j}}(\psi_j) = e^{-i\pi N_j \eta_j},
\]

\[
c_{-N_{j}}(\psi_j) = e^{i\pi N_j \eta_j} = e^{i\pi c_{-N_{j}}(\eta_j)}.
\]

With this we can easily verify (24) and the first assertion follows with Lemma 4.6.

Now suppose \( p \) is even and \( \lambda \geq 1 \). Furthermore, let \( T = T_{j+1}^{p_1} \cdots T_{j+1} \) be the translation operator of \( \psi_j^{\lambda -1p/2} \). Then \( T \) is also the translation operator of \( \psi_j^{\lambda p} \) and the one of \( \psi_j^{\lambda p+1} \) is \( T' = T_{j+1}^{p_1} \). Let \( r \in \mathbb{R} \) such that \( T(f)(x) = f(x-r) \). We obtain

\[
c_{N}(\psi_j^{\lambda -1p/2}) = e^{-i\pi N_j c_{N}(\eta_j)} \left( e^{i\pi N_j} \right), \quad c_{-N}(\psi_j^{\lambda -1p/2}) = e^{+i\pi N_j c_{N}(\eta_j)} \left( e^{-i\pi N_j} \right),
\]

\[
c_{N}(\psi_j^{\lambda p}) = e^{-i\pi N_j c_{N}(\eta_j)} \left( e^{i\pi N_j} \right), \quad c_{-N}(\psi_j^{\lambda p}) = e^{+i\pi N_j c_{N}(\eta_j)} \left( e^{-i\pi N_j} \right),
\]

\[
\psi_j^{\lambda p} = e^{-i\pi N_j c_{N}(\eta_j)} \left( e^{i\pi N_j} \right), \quad \psi_j^{\lambda p+1} = e^{+i\pi N_j c_{N}(\eta_j)} \left( e^{-i\pi N_j} \right).
\]

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where $N$ is defined as in Lemma 4.7. Equation (26) follows easily. Furthermore,

$$
\begin{align*}
    c_K(\psi_j^{\lambda-1/2p/2}) &= e^{-iKr}c_K (\eta_j^{\lambda-1/2p/2}), \\
    c_{-K}(\psi_j^{\lambda-1/2p/2}) &= e^{+iKr}c_K (\eta_j^{\lambda-1/2p/2}), \\
    c_K(\psi_j^{\lambda-1/2p+1}) &= e^{-iKr}C_K^{\lambda-1/2p+1} c_K (\eta_j^{\lambda-1/2p+1}), \\
    c_{-K}(\psi_j^{\lambda-1/2p+1}) &= e^{+iKr}C_K^{\lambda-1/2p+1} c_K (\eta_j^{\lambda-1/2p+1}).
\end{align*}
$$

By definition of $K$ and $N_j$ and since $p$ is even we have that

$$
\frac{K}{N_j^{\lambda+1}} = \frac{N_j + (p+2)N_j^{\lambda-\lambda}}{2N_j^{\lambda-\lambda}} \in \mathbb{Z}
$$

and hence, condition (27) is easy to verify. Moreover,

$$
\begin{align*}
    c_M(\psi_j^{\lambda p}) &= e^{-iMr}c_M (\eta_j^{\lambda p}), \\
    c_{-M}(\psi_j^{\lambda p}) &= e^{+iMr}c_M (\eta_j^{\lambda p}), \\
    c_M(\psi_j^{\lambda p+1}) &= e^{-iMr}C_M^{\lambda p+1} c_M (\eta_j^{\lambda p+1}), \\
    c_{-M}(\psi_j^{\lambda p+1}) &= e^{+iMr}C_M^{\lambda p+1} c_M (\eta_j^{\lambda p+1}),
\end{align*}
$$

and by definition of $M$ and since $p$ is even we get

$$
\frac{M}{N_j^{\lambda+1}} = \frac{N_j + (p+1)N_j^{\lambda-\lambda}}{2N_j^{\lambda-\lambda}} = \left( k + \frac{p+1}{2} \right) \in \mathbb{Z} + 1/2.
$$

Thus, relation (25) holds. This completes the proof. 

In view of (28), we define for notational convenience

$$
T_j^{\lambda p} := T_j^{n \mod 2} T_j^{ \lfloor k \mod 2 \rfloor} \cdots T_j^{ \lfloor \frac{p-1}{2} \mod 2 \rfloor} T_{j+1}.
$$

(29)

Now, we apply the preceding results to our functions $\sigma_n$ and $\sigma_m$, which we proved to be time optimal.

**Corollary 4.8** Let $j \in \mathbb{N}_0$, $0 \leq \lambda \leq j$, $0 \leq p \leq 2\lambda - 1$ and define

$$
\begin{align*}
    \phi_j &= \sigma_{N_j}, \\
    \psi_j^{\lambda p} &= T_j^{\lambda p} \sigma_{N_j^{\lambda p+1}}.
\end{align*}
$$

The functions $\phi_j$ generate a multiresolution analysis, the functions $\psi_j = \psi_j^{00}$ are wavelets corresponding to this MRA and the functions $\psi_j^{\lambda p}$ are wavelet packets.

**Proof:** The assertion follows immediately from Theorem 4.8.

Now, as we have constructed a wavelet packet basis, the immediate question arises whether it is stable.

**Lemma 4.10** Let $\mathcal{D}$ be a wavelet decomposition of $L^2_{2\pi}$ generated by the time optimal polynomials from Corollary 4.9. The functions

$$
\{ T_S^\alpha \psi_s | s = 0, \ldots, 2N^\alpha - 1, \alpha \in \mathcal{D} \}
$$

form a Riesz basis for $L^2_{2\pi}$ iff the numbers $N^\alpha$, $\alpha \in \mathcal{D}$, or equivalently the dimensions of the spaces $W^\alpha$, $\alpha \in \mathcal{D}$, are bounded.
Proof: We have to consider the quotients $D^\alpha/C^\alpha$ where $C^\alpha$ and $D^\alpha$ are defined in (22), (23). By definition of $\psi^\alpha$, for $\alpha \neq (j,0,-1)$ and $\alpha \neq (j,j,p)$, we obtain

$$C^\alpha = 2^\alpha \min\left\{ \sin \frac{k\pi}{N^\alpha + 2} \mid k = 2, \ldots, N^\alpha + 1 \right\} \cup \left\{ 2 \sin \frac{\pi}{N^\alpha + 2} \right\}$$
$$D^\alpha = 2^\alpha .$$

We obtain

$$\frac{D^\alpha}{C^\alpha} = \frac{1}{\sin \frac{2\pi}{N^\alpha + 2}} .$$

The case $\alpha = (j,0,-1)$ is analogous. If $\alpha = (j,j,p)$, i.e. $N^\alpha = c$, and $c = 2q - 1$ is odd then

$$D^\alpha = 2^\alpha \max\{ \sin \frac{k\pi}{2q + 1} \mid k = 1, \ldots, 2q \} = 2^\alpha \sin \frac{q\pi}{2q + 1}$$
$$\frac{D^\alpha}{C^\alpha} = \sin \frac{q\pi}{2q + 1} .$$

If $c$ is even then formulas (30), (31) apply, again. Altogether, the quotients $D^\alpha/C^\alpha$ are bounded if and only if the numbers $N^\alpha \equiv N_{j,\lambda}$ are bounded.

From the last lemma it follows in particular that in case of a wavelet decomposition, i.e. if no wavelet packets are involved, the generated basis is not a Riesz basis.

Theorem 4.11. Given a wavelet packet decomposition $D$ of $L^2_{2\pi}$ generated by the time optimal polynomials from Corollary 4.9, the following two conditions exclude each other:

1. $u := \sup_{\beta \in D} U_{2\pi}(\psi^\alpha) < \infty$.
2. The basis $B$ (see (21)) generated by $D$ is a Riesz basis.

Proof: Let $\alpha = (j,\lambda,p)$. By (19) we obtain

$$U_{2\pi}^2(\psi^\alpha) = \left( \frac{(N_{j,\lambda})(N_{j,\lambda} + 1)}{3} + N_j^{\lambda p} N_j^{\lambda p + 1} \right) \tan^2 \frac{\pi}{N_{j,\lambda} + 2} - \frac{1}{2} .$$

Now suppose $u < \infty$. This implies that in particular the term

$$N_j^{\lambda p} N_j^{\lambda p + 1} \tan^2 \frac{\pi}{N_{j,\lambda} + 2}$$

is bounded. But this is not possible if $N_{j,\lambda}$ is bounded since $N_j^{\lambda p}$ is always unbounded. With Lemma 4.10 it immediately follows that $B$ cannot be a Riesz basis.

On the other hand, suppose that $B$ is a Riesz basis, Lemma 4.10 then implies that $N_{j,\lambda}$ is bounded. It immediately follows that $U_{2\pi}(\psi^\alpha)$ is unbounded.

We remark that in contrast to Theorem 4.11 there are examples of periodic wavelets constructed in [6] and [13] that have uniformly bounded uncertainty products and yet form a Riesz basis of $L^2_{2\pi}$. However, concerning the trigonometric wavelets in [13] the overlap of the frequency support of the basis functions of neighboring wavelet-packet spaces grows proportional with the highest frequency of the corresponding wavelet (packet) space when the uncertainty product is bounded [13, Theorem 3]. So one might conjecture that adding a third condition of non-growing frequency overlap in Theorem 4.11 will give a general result in the sense that all three conditions cannot hold simultaneously.
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