Wavelet transforms associated to group representations and functions invariant under symmetry groups

Holger Rauhut

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Abstract

We study the wavelet transform of functions invariant under a symmetry group, where the wavelet transform is associated to an irreducible unitary group representation. Among other results a new inversion formula and a new covariance principle are derived. As main examples we discuss the continuous wavelet transform and the short time Fourier transform of radially symmetric functions on $\mathbb{R}^d$.

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1 Introduction

In the recent years a lot of research activities were dedicated to wavelet analysis and time-frequency analysis. The outcome of these theories could be successfully applied in various areas such as signal and image processing, the numerical solution of partial differential equations and wireless communication just to mention a few. This paper is dedicated to the wavelet and time-frequency-analysis of functions which are invariant under certain symmetry groups. This aspect of time-frequency-analysis and wavelet analysis was not treated thoroughly in the literature before, at least according to my knowledge.

Suppose a function on $\mathbb{R}^d$ possesses some symmetries, i.e. is invariant under some symmetry group. Our main example will be radial symmetry or in other words invariance under $SO(d)$, but also other symmetry groups such as finite reflection groups may be considered. Of course, it is possible to do standard wavelet and time-frequency analysis with such a function. For example we may expand it in terms of certain building blocks which will be translates and dilates of a single function in case of wavelet analysis and which consist of modulations and translations of a single function in case of time-frequency analysis. Hereby the expansion can be discrete but also a continuous superposition such as in the inversion formula for the continuous wavelet transform or for the short time Fourier transform (STFT). However, in both examples the building blocks do not in general obey the same symmetry properties as the analyzed function. For example, even if we start with a radial symmetric wavelet, a translated version of it will no longer be radial. So one might ask the question whether it is possible to make use of the symmetry properties. In other words, is it possible to construct the building blocks in such a way that they all possess certain symmetry properties? And can this be done such that all building blocks are derived from one single function similarly as in standard wavelet and time-frequency analysis? This paper presents a very natural approach to this problem. We will only cover the continuous wavelet transform (CWT) and the (continuous) STFT. Their discrete counterparts, i.e. the construction of invariant wavelet frames and invariant Gabor frames, will be treated in a subsequent paper. A generalization of the Feichtinger-Gröchenig theory [7, 8, 10] will be developed to solve the discretization problem. In the special case of radial functions on $\mathbb{R}^3$ the concept of multiresolution analysis could already be successfully applied to the construction of radial wavelets [15].

The CWT and the STFT are closely connected to the representation theory of the similitude group of $\mathbb{R}^d$ and the Heisenberg group, respectively [1, 11]. It is therefore possible to treat both transforms simultaneously in the general framework of representation theory of locally compact groups. This is the reason why I have chosen to investigate the problem in this abstract setting and then specialize to the examples afterwards.

In order to involve a symmetry group $\mathcal{A}$ in this abstract setting we need an action on the group $\mathcal{G}$ whose representation coefficients $V_g f(x) = \langle f, \pi(x)g\rangle_{\mathcal{H}}$ give the corresponding wavelet transform where π acting on the Hilbert space $\mathcal{H}$ denotes the representation under consideration. (Here we use the term wavelet transform also in the general context). It is reasonable to assume that $\mathcal{A}$ is a compact automorphism group of $\mathcal{G}$. Moreover, we require that $\mathcal{A}$ has a representation σ on the same Hilbert space $\mathcal{H}$ (usually $L^2(\mathbb{R}^d)$ in our examples). As a basic assumption all representations$\pi_A := \pi \circ A, A \in \mathcal{A}$ have to be equivalent to π with intertwining operators $\sigma(A)$, see formula (3.1). The invariant elements of $\mathcal{H}$ are those that satisfy $\sigma(A)f = f$ for all $A \in \mathcal{A}$. It turns out that $V_g f$ is invariant under $\mathcal{A}$ if f, g are invariant and can hence be viewed as a function on the orbits $\mathcal{A}(x), x \in \mathcal{G}$. The space $\mathcal{K}$ of all orbits has a structure called a hypergroup (a generalization of a group) and in an easy way starting from π we construct a representation $\tilde{\pi}$ of $\mathcal{K}$ on $\mathcal{H}_{\mathcal{A}}$, the Hilbert space of all invariant elements of $\mathcal{H}$. The operators $\tilde{\pi}(x)$ map $\mathcal{H}_{\mathcal{A}}$ into $\mathcal{H}_{\mathcal{A}}$ and thus it seems natural to take the elements $\tilde{\pi}(x)g$ as new building blocks. In fact, it holds $V_g f = \langle f, \tilde{\pi}(x)g\rangle_{\mathcal{H}_{\mathcal{A}}}$ for invariant f, g. In our special cases with radial symmetry the scalar product on $\mathcal{H}_{\mathcal{A}} = L^2_{rad}(\mathbb{R}^d)$ (the space of radial L^2-functions) can actually be computed by an integral over the positive half line, so this formula means a reduction in complexity. Moreover, we prove as a main result another inversion formula for the wavelet transform where an invariant element is represented by a continuous superposition of the invariant building blocks $\tilde{\pi}(x)g$. Also a covariance principle, which involves a generalized translation coming from the hypergroup $\mathcal{K}$, is shown for the wavelet transform of invariant elements.

The outline of the paper is the following. We start by discussing briefly the CWT of functions with radial symmetry as a motivating example. Afterwards we present the abstract general approach including the main results. In section 4 we continue the discussion of the continuous wavelet transform of radial function thereby illustrating the general theory. It is worth noting that Rösler [16] and independently Trimeche [17] had already introduced this transform (writing it is a transform on the positive halfline) in the context of hypergroup theory. As a second example we discuss the STFT of radially symmetric functions. Actually this example was the original motivation for the investigations contained in this paper. There have been already other approaches for a radial STFT [3, 4, 17]. Unfortunately the transforms introduced in these papers lack some important properties. For example one can show that the Wigner-Bessel transform introduced in [4] has unbounded inverse (which however is not proven in [4]) and the transform in [3] makes use of a nonlinear modulation which seems a bit strange. As a last example we discuss briefly the situation when $SO(d)$ in both previous examples is replaced by a finite reflection group.

2 A motivating example

We start with the basic example of the continuous wavelet transform of radial functions. Let $L^2(\mathbb{R}^d)$ denote the Hilbert space of all complex-valued square-integrable functions on $\mathbb{R}^d$ with the usual norm and scalar product, denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$. We introduce the following unitary operations on $L^2(\mathbb{R}^d)$, the translation $T_x f(y) := f(y-x), x \in \mathbb{R}^d$, the dilation $D_a f(y) := a^{-d/2} f(y/a), a \in \mathbb{R}_+^* := (0,\infty)$ and the rotation $U_R f(y) = f(R^{-1}y), R \in SO(d)$, where $SO(d)$ denotes the special orthogonal group in dimension d. For a function $\psi \in L^2(\mathbb{R}^d)$ (called the wavelet) the continuous wavelet transform on $\mathbb{R}^d$ is defined by

$$\begin{aligned} V_\psi f(x,a,R) &:= \langle f, T_x D_a U_R \psi\rangle \\ &= a^{-d/2}\int_{\mathbb{R}^d} f(y)\overline{\psi(a^{-1}R^{-1}(y-x))}dy, \quad x\in\mathbb{R}^d, a\in\mathbb{R}_+^*, R\in SO(d). \end{aligned} \tag{2.1}$$

A wavelet $\psi \neq 0$ is called admissible if the wavelet transform $V_\psi f$ is square integrable for all $f \in L^2(\mathbb{R}^d)$, i.e. if

$$\int_{\mathbb{R}^d}\int_0^\infty\int_{SO(d)} |V_\psi f(x,a,R)|^2 dR \frac{da}{a^{d+1}} dx < \infty \quad \text{for all } f \in L^2(\mathbb{R}^d)$$

where dR denotes the normalized Haar measure on $SO(d)$. It is well-known [1] that this condition is satisfied iff

$$c_\psi := \int_{\mathbb{R}^d} \frac{|\hat{\psi}(\xi)|^2}{|\xi|^d} d\xi < \infty \tag{2.2}$$

where $\hat{\psi}$ denotes the Fourier transform of ψ, i.e. $\hat{\psi}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \psi(x) e^{-ix\cdot\xi} dx$ and $|\cdot|$ denotes the Euclidean norm.

We are interested in functions f that are radially symmetric, i.e. $f(R^{-1}x) = f(x)$ for all $R \in SO(d)$. For a radial f there exists a function f_0 on $\mathbb{R}_+ := [0, \infty)$ such that $f(x) = f_0(|x|)$. A change to polar coordinates shows that $\int_{\mathbb{R}^d} |f(x)|^2 dx = |S^{d-1}| \int_0^\infty |f_0(r)|^2 r^{d-1} dr$ where $|S^{d-1}| = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ denotes the surface area of the sphere S^{d-1}. Hence $L^2_{rad}(\mathbb{R}^d)$ (the subspace of $L^2(\mathbb{R}^d)$ of radial functions) is isometrically isomorphic to $L^2(\mathbb{R}_+, \mu_d)$ where $d\mu_d(r) = |S^{d-1}| r^{d-1} dr$.

If ψ and f are radial then a simple calculation shows

$$V_\psi f(S^{-1}x, a, R) \;=\; V_\psi f(x, a, Id) \quad \text{for all } S, R \in SO(d).$$

With radial ψ we denote the restriction of V_ψ to $L^2_{rad}(\mathbb{R}^d)$ by $\tilde{V}_\psi$. Of course, $\tilde{V}_\psi$ depends only on $a \in R^*_+$ and $|x| \in \mathbb{R}^d$ and may therefore be interpreted as a function on $\mathbb{R}_+ \times \mathbb{R}^*_+$. Using Fubini's theorem we may come up with the following formula

$$\begin{aligned} \tilde{V}_\psi f(x, a) &= \int_{SO(d)} V_\psi f(S^{-1}x, a, Id) dS \\ &= a^{-d/2} \int_{\mathbb{R}^d} f(y) \overline{\int_{SO(d)} \psi(a^{-1}(y - S^{-1}x)) dS} dy. \end{aligned} \tag{2.3}$$

Denoting

$$\tau_x g(y) \;:=\; \int_{SO(d)} g(y - S^{-1}x) dS \tag{2.4}$$

we have $\tilde{V}_\psi f(x, a) \;=\; \langle f, \tau_x D_a \psi \rangle$. The operation τ_x is called a generalized translation. It is easy to see that τ_x preserves radiality. Using Weil's formula (see for instance [9, Theorem 2.49]) for the Haar-measure on $SO(d)$ one deduces the following formula for $\tau_x g$ involving the corresponding function g_0 on $\mathbb{R}_+$,

$$\begin{aligned} \tau_x g(y) &= \frac{1}{|S^{d-1}|} \int_{S^{d-1}} g(y - |x|\xi) dS(\xi) \\ &= \frac{|S^{d-2}|}{|S^{d-1}|} \int_{-1}^{1} g_0(\sqrt{s^2 - 2rst + r^2})(1 - t^2)^{(d-3)/2} dt, \quad r = |x|, s = |y|. \end{aligned}$$

Clearly, $\tau_x g(y) = \tau_y g(x)$ and $\tau_x g(y)$ depends only on $|x|$ and $|y|$. Hence, it makes sense to use the notation $\tau_s g_0(r), r, s \in \mathbb{R}_+^*$ for the corresponding operation on function on $\mathbb{R}_+^*$. Since $L^2_{rad}(\mathbb{R}^d)$ and $L^2(\mathbb{R}_+, \mu_d)$ are isometrically isomorphic we can express the wavelet transform also by an integral over $\mathbb{R}_+$, i.e.

$$\tilde{V}_\psi f(s,a) \;=\; |S^{d-1}| \int_0^\infty f(r)\overline{\tau_s D_a \psi_0(r)} r^{d-1} dr, \quad s \in \mathbb{R}_+, a \in \mathbb{R}_+^*. \tag{2.5}$$

The generalized translation τ is actually deeply linked to hypergroup theory [2, 13], more precisely to the so-called Bessel-Kingman hypergroup. And in fact the transform (2.5) is essentially what Margit Rösler [16] and independently Trimeche [17] introduced as the wavelet transform on the Bessel-Kingman hypergroup.

Starting from the inversion formula for the wavelet transform and using a similar trick as in (2.3) we derive a second inversion formula for the wavelet transform of radial functions

$$f(t) \;=\; |S^{d-1}| \int_{\mathbb{R}_+} \int_{\mathbb{R}_+^*} \tilde{V}_\psi f(s,a) \tau_s D_a \psi(t) \frac{da}{a^{d+1}} r^{d-1} dr \quad \text{a.e.} \tag{2.6}$$

where ψ is admissible and normalized such that $c_\psi = 1$. Since this inversion formula will also follow from a general theorem derived in the next section we skip the details of its proof at this place.

The formula (2.6) states in particular that we may represent a radial function as a continuous superposition of the radial(!) functions $\tau_s D_a \psi, s \in \mathbb{R}_+, b \in \mathbb{R}_+^*$. The question arises whether one can discretize this formula in order to have a radial function represented as a linear combination of radial functions which are all derived from a single function ψ in a wavelet-like way. In the special case of $\mathbb{R}^3$ this problem was recently solved via the concept of multiresolution analysis [15].

3 Group representations and automorphism groups

We turn over now to the general abstract setting. Let $\mathcal{G}$ be a locally compact group and $\mathcal{A}$ be a *compact* automorphism group (symmetry group) of $\mathcal{G}$, such that $\mathcal{A}$ acts continuously on $\mathcal{G}$, i.e. the mapping $\mathcal{G} \times \mathcal{A} \to \mathcal{G}, (x, A) \mapsto A(x)$ is continuous (where the product topology is taken on $(\mathcal{G}, \mathcal{A})$). We denote the left Haar measures on $\mathcal{G}$ and $\mathcal{A}$ by μ and ν, where ν is assumed to be normalized. However, we usually will write dx and dA in integrals. The modular function on $\mathcal{G}$ is denoted by Δ. (Since $\mathcal{A}$ is compact it is unimodular and no modular function is needed.) It is standard to show (making use of the compactness of $\mathcal{A}$) that the Haar measure μ and the modular function Δ are invariant under $\mathcal{A}$. Given a function F on $\mathcal{G}$ and $A \in \mathcal{A}$ we define $F_A(x) := F(A^{-1}x)$.

Further assume that we have given an irreducible unitary (strongly continuous) representation π of $\mathcal{G}$ on a Hilbert space $\mathcal{H}$ and a unitary representation σ (not necessarily irreducible) of $\mathcal{A}$ on the same Hilbert space $\mathcal{H}$ such that

$$\pi(A(x))\sigma(A) \;=\; \sigma(A)\pi(x). \tag{3.1}$$

In other words, we require that the representations $\pi_A := \pi \circ A$ are all unitarily equivalent to π and that the intertwining operators $\sigma(A)$ form a representation of $\mathcal{A}$. The condition (3.1) will be essential in what follows. If for example $\mathcal{A}$ is a compact subgroup of $\mathcal{G}$ acting by inner automorphisms and $\sigma = \pi_{|\mathcal{A}}$ then (3.1) is trivial to check.

For $f \in \mathcal{H}$ we let $f_A = \sigma(A)f$ and $\mathcal{H}_{\mathcal{A}} \;:=\; \{f \in \mathcal{H} \mid \sigma(A)f \;=\; f \text{ for all } A \in \mathcal{A}\}$, the closed(!) subspace of invariant elements. We always assume that $\mathcal{H}_{\mathcal{A}}$ is not trivial. The wavelet transform or voice transform is defined by

$$V_g(f)(x) \;:=\; \langle f, \pi(x)g\rangle.$$

It maps $\mathcal{H}$ into $C^b(\mathcal{G})$, the space of bounded continuous functions on $\mathcal{G}$. With an element $g \in \mathcal{H}_{\mathcal{A}}$ we denote by $\tilde{V}_g$ the restriction of V_g to $\mathcal{H}_{\mathcal{A}}$. Further we define

$$C^b_{\mathcal{A}}(\mathcal{G}) \;:=\; \{F \in C^b(\mathcal{G}), F_A = F \text{ for all } A \in \mathcal{A}\}.$$

Lemma 3.1. *Suppose that (3.1) holds.*

(a) For $f, g \in \mathcal{H}$ we have $(V_g f)_A(x) = V_{g_A} f_A(x)$.

(b) Consequently with $g \in \mathcal{H}_{\mathcal{A}}$, $\tilde{V}_g$ maps $\mathcal{H}_{\mathcal{A}}$ into $C^b_{\mathcal{A}}(\mathcal{G})$.

(c) For $x \in \mathcal{G}$ define the operator

$$\tilde{\pi}(x) \;:=\; \int_{\mathcal{A}} \pi(Ax)\,dA$$

where the integral is understood weakly, i.e. $\langle f, \tilde{\pi}(x)g\rangle \;=\; \int_{\mathcal{A}}\langle f, \pi(Ax)g\rangle dA$ for all $f, g \in \mathcal{H}$. Then (with $g \in \mathcal{H}_{\mathcal{A}}$) it holds

$$\tilde{V}_g f(x) \;=\; \langle f, \tilde{\pi}(x)g\rangle_{\mathcal{H}_{\mathcal{A}}}.$$

(d) The operators $\tilde{\pi}(x), x \in \mathcal{G}$ do not depend on the choice of x from the orbit $\mathcal{A}(x'), x' \in \mathcal{G}$, i.e. $\tilde{\pi}(Ax) = \tilde{\pi}(x)$ for all $A \in \mathcal{A}$, and $\tilde{\pi}(x)$ maps $\mathcal{H}_{\mathcal{A}}$ into $\mathcal{H}_{\mathcal{A}}$ for all $x \in \mathcal{G}$.

Proof: (a) Using (3.1) we obtain

$$\begin{aligned} V_g f(A^{-1}x) \;&=\; \langle f, \pi(A^{-1}x)g\rangle \;=\; \langle f, \sigma(A^{-1})\pi(x)\sigma(A)g\rangle \\ &=\; \langle \sigma(A)f, \pi(x)\sigma(A)g\rangle \;=\; V_{g_A} f_A(x). \end{aligned}$$

(b) If $g_A = g$ and $f_A = f$ then as a consequence of (a) clearly $V_g f(A^{-1}x) \;=\; V_g f(x)$.

(c) Using (b) we have for $f, g \in \mathcal{H}_A$

$$V_g f(x) \;=\; \int_{\mathcal{A}} V_g f(Ax) dA \;=\; \int_{\mathcal{A}} \langle f, \pi(Ax) g \rangle dA$$

which is nothing else than (c).

(d) Using the translation invariance of the Haar measure of $\mathcal{A}$ we immediately get $\tilde{\pi}(Bx) = \tilde{\pi}(x)$ for all $B \in \mathcal{A}$. Furthermore for $f \in \mathcal{H}_{\mathcal{A}}$, using (3.1) we get

$$\begin{aligned} \sigma(B)\tilde{\pi}(x) f &= \sigma(B) \int_{\mathcal{A}} \pi(Ax) f dA \;=\; \int_{\mathcal{A}} \sigma(B)\sigma(A)\pi(x)\sigma(A)^{-1} f dA \\ &= \int_{\mathcal{A}} \sigma(BA)\pi(x)\sigma(BA)^{-1} f dA \;=\; \int_{\mathcal{A}} \pi(BAx) f dA \;=\; \tilde{\pi}(x) f \end{aligned}$$

where all expressions are understood in a weak sense. Notice that we have made use of the invariance of f in the third equality. Hence, $\tilde{\pi}(x) f \in \mathcal{H}_{\mathcal{A}}$. ■

For the following we need to recall some facts about convolution of measures. We denote by $M(\mathcal{G})$ the space of all bounded Radon measures on $\mathcal{G}$, i.e the dual space of the space $C_0(\mathcal{G})$ of continuous functions on $\mathcal{G}$ vanishing at infinity. A function $G \in L^1(\mathcal{G})$ can be identified with an element μ_G of $M(\mathcal{G})$ by setting $\mu_G(F) = \int_{\mathcal{G}} F(x) G(x) d\mu(x)$ and $L^1(\mathcal{G})$ can be viewed as a closed subspace of $M(\mathcal{G})$. With the convolution

$$\tau * \rho(F) \;:=\; \int_{\mathcal{G}} \int_{\mathcal{G}} F(xy) d\tau(x) d\rho(y), \quad \tau, \rho \in M(\mathcal{G}), F \in C_0(\mathcal{G})$$

and the involution $\tau^*(F) := \overline{\int_{\mathcal{G}} \overline{F(x^{-1})} d\tau(x)}$ $M(\mathcal{G})$ becomes a Banach-$*$-algebra which contains $L^1(\mathcal{G})$ as a closed subalgebra (actually as a two-sided ideal). The formula for the convolution of two functions $F, G \in L^1(\mathcal{G})$ reads $F * G(x) = \int_{\mathcal{G}} F(y) G(y^{-1}x) d\mu(y)$.

For a measure τ we denote the action of $A \in \mathcal{A}$ on τ by $\tau_A(F) = \tau(F_{A^{-1}}), F \in C_0(\mathcal{G})$. The closed subspace of $M(\mathcal{G})$ of invariant bounded measures will be denoted by $M_{\mathcal{A}}(\mathcal{G})$, i.e. $M_{\mathcal{A}}(\mathcal{G}) := \{\tau \in M(\mathcal{G}), \tau_A = \tau$ for all $A \in \mathcal{A}\}$. We remark that the Haar measure μ of $\mathcal{G}$ is contained in $M_{\mathcal{A}}(\mathcal{G})$. Another invariant measure of interest is given for $x \in \mathcal{G}$ by

$$\epsilon_{\mathcal{A}x}(F) = \int_{\mathcal{A}} F(Ax) dA.$$

We call $\epsilon_{\mathcal{A}x}$ invariant Dirac measure because for an invariant function F we obviously have $\epsilon_{\mathcal{A}x}(F) = F(x)$. An invariant measure (function) can also be identified with a measure (function) on the orbits $\mathcal{A}x := \{A(x), A \in \mathcal{A}\}$. The space $\mathcal{K} := \mathcal{A}(\mathcal{G})$ of all orbits becomes a topological space in a natural way by defining a set $U \subset \mathcal{K}$ open iff U viewed as a subset of $\mathcal{G}$ that satisfies $U = \mathcal{A}(U)$ is open in the topology of $\mathcal{G}$. It makes sense to use the symbol

$M(\mathcal{K})$ instead of $M_{\mathcal{A}}(\mathcal{G})$. The space $\mathcal{K}$ has a canonical measure m inherited from the Haar-measure of $\mathcal{G}$, i.e. for an invariant function in $L^1(\mathcal{G})$, $\int_{\mathcal{K}} F(\mathcal{A}x)dm(\mathcal{A}x) = \int_{\mathcal{G}} F(x)d\mu(x)$ where we use the same symbol for a function on $\mathcal{K}$ and the corresponding invariant one on $\mathcal{G}$. The space $L^1(\mathcal{K}) := L^1(\mathcal{K}, m)$ is isomorphic to $L^1_{\mathcal{A}}(\mathcal{G})$, the subspace of $L^1(\mathcal{G})$ of invariant functions.

Let us now collect some properties of the convolution of invariant functions.

Lemma 3.2. *(a) $M_{\mathcal{A}}(\mathcal{G}) = M(\mathcal{K})$ is a closed subalgebra of $M(\mathcal{G})$, i.e. the convolution of two invariant measures is again invariant. It follows that $L^1_{\mathcal{A}}(\mathcal{G}) = L^1(\mathcal{K})$ is a closed subalgebra of $L^1(\mathcal{G})$.*

(b) For $x, y \in \mathcal{G}, F \in C^b(\mathcal{G})$ let

$$\mathcal{T}_y F(x) := \int_{\mathcal{A}} F(A(y)x)dA, \qquad \mathcal{L}_y F(x) := \int_{\mathcal{A}} F(A(y^{-1})x)dA = \mathcal{T}_{y^{-1}}F(x).$$

If $F \in C^b_{\mathcal{A}}(\mathcal{G})$ then $\mathcal{T}_y F, \mathcal{L}_y F \in C^b_{\mathcal{A}}(\mathcal{G})$ for all $y \in \mathcal{G}$ and both expressions depend only on the orbit $\mathcal{A}y$. Moreover, it holds

$$\mathcal{L}_y F(x) \;=\; \epsilon_{\mathcal{A}y} * F(x) \;=\; \epsilon_{\mathcal{A}y^{-1}} * \epsilon_{\mathcal{A}x}(F). \tag{3.2}$$

(c) If $F, G \in L^1(\mathcal{K})$ then

$$F * G(x) \;=\; \int_{\mathcal{K}} F(y)\mathcal{L}_y G(x)dm(y). \tag{3.3}$$

(d) If $\tau, \rho \in M(\mathcal{K})$ and $F \in C_0^{\mathcal{A}}(\mathcal{G}) = \{F \in C_0(\mathcal{G}), F_A = F \text{ for all } A \in \mathcal{A}\}$ then

$$\tau * \rho(F) \;=\; \int_{\mathcal{K}}\int_{\mathcal{K}} \mathcal{T}_x F(y)d\tau(y)d\rho(x).$$

(e) Define an involution on $\mathcal{K}$ by $(\mathcal{A}x)^{\tilde{}} := \mathcal{A}(x^{-1})$. Then $(\mathcal{K}, *, \tilde{})$ is a hypergroup.

Proof: (a) For $\tau, \rho \in M_{\mathcal{A}}(\mathcal{G}), F \in C_0(\mathcal{G})$ and $A \in \mathcal{A}$ we obtain

$$\begin{aligned}\tau * \rho(F_{A^{-1}}) &= \int_{\mathcal{G}}\int_{\mathcal{G}} F(A(xy))d\tau(x)d\rho(y) = \int_{\mathcal{G}}\int_{\mathcal{G}} F(A(x)A(y))d\tau(x)d\rho(y)\\ &= \int_{\mathcal{G}}\int_{\mathcal{G}} F(xy)d\tau(x)d\rho(y) = \tau * \rho(F).\end{aligned} \tag{3.4}$$

The assertions in (b) are verified with easy computations. Note that the expressions in (3.2) make sense also for $F \in C^b(\mathcal{G})$ since the measures $\epsilon_{\mathcal{A}x}$ and $\epsilon_{\mathcal{A}y^{-1}} * \epsilon_{\mathcal{A}x}$ have compact support. For (c) we use (a) and Fubini's theorem in order to obtain

$$\begin{aligned}(F*G)(x) &= \int_{\mathcal{A}}(F*G)(Ax)dA = \int_{\mathcal{G}}F(y)\int_{\mathcal{A}}G(y^{-1}Ax))dAdx \\ &= \int_{\mathcal{K}}F(y)\mathcal{L}_yG(x)dm(y)\end{aligned}$$

and (d) is deduced similarly. For (e) we refer to [13, Theorem 8.3A.]. (Jewett uses the term convo instead of hypergroup.) ■

Remark 3.1. *For reasons of length we will not go into details on hypergroups which are generalizations of group algebras. The interested reader is referred to [2, 13]. Although this paper can also be read without knowing much about hypergroups this theory is very present in the background and a lot of motivation for this paper took essentially its roots in hypergroup theory.*

Usually the operators $\mathcal{T}_y, \mathcal{L}_y$ are called generalized translation operators, $\mathcal{L}_y$ being the generalized left translation.

It is well-known [9] that the representation π of $\mathcal{G}$ extends to a non-degenerate $*$-representation of $M(\mathcal{G})$ by letting

$$\pi(\mu) := \int_{\mathcal{G}}\pi(x)d\mu(x)$$

where the integral is understood in a weak sense. In other words $\pi(\tau*\rho) = \pi(\tau)\pi(\rho)$ and $\pi(\tau^*) = \pi(\tau)^*$. Further observe that $\pi(\epsilon_{\mathcal{A}x}) = \tilde{\pi}(x)$.

Now we are ready to state a covariance principle for $\tilde{V}_g$.

Theorem 3.3. *Let $f, g \in \mathcal{H}_{\mathcal{A}}, y \in \mathcal{G}$. Then*

$$\tilde{V}_g(\tilde{\pi}(y)f) = \mathcal{L}_y\tilde{V}_gf. \tag{3.5}$$

<u>**Proof:**</u> Obviously, it holds $(\epsilon_{\mathcal{A}y})^* = \epsilon_{\mathcal{A}y^{-1}}$. We therefore obtain

$$\begin{aligned}\tilde{V}_g(\tilde{\pi}(y)f)(x) &= \langle\pi(\epsilon_{\mathcal{A}y})f, \pi(\epsilon_{\mathcal{A}x})g\rangle = \langle f, \pi(\epsilon_{\mathcal{A}y^{-1}})\pi(\epsilon_{\mathcal{A}x})g\rangle = \langle f, \pi(\epsilon_{\mathcal{A}y^{-1}}*\epsilon_{\mathcal{A}x})g\rangle \\ &= \epsilon_{\mathcal{A}y^{-1}}*\epsilon_{\mathcal{A}x}(\tilde{V}_gf) = \mathcal{L}_y\tilde{V}_gf(x).\end{aligned}$$

■

For completeness we state a Lemma which might be the starting point for further generalizations of this paper.

Lemma 3.4. *The mapping $\tilde{\pi} : \mathcal{K} \to \mathcal{B}(\mathcal{H}_{\mathcal{A}})$, where $\mathcal{B}(\mathcal{H}_{\mathcal{A}})$ denotes the space of all bounded operators on $\mathcal{H}_{\mathcal{A}}$, is an irreducible representation of the hypergroup $(\mathcal{K}, *, \tilde{\ })$.*

Proof: It is proven in Lemma 3.1 that $\tilde{\pi}(x)$ is a mapping from $\mathcal{H}_{\mathcal{A}}$ into $\mathcal{H}_{\mathcal{A}}$. The properties $\pi(\epsilon_{\mathcal{A}x} * \epsilon_{\mathcal{A}y}) = \pi(\epsilon_{\mathcal{A}x})\pi(\epsilon_{\mathcal{A}y})$ and $\pi(\epsilon_{(\mathcal{A}x)^{\check{}}}) = \pi(\epsilon^*_{\mathcal{A}x}) = \pi(\epsilon_{\mathcal{A}x})^*$ follow from the fact that π generates a $*$-representation of $M(\mathcal{G})$. For the proof of the irreducibility of $\tilde{\pi}$ the irreducibility of π is used. We skip the details since this fact will not be used later on. ■

For the following we make the additional assumption that π is square integrable, which means that there exists a vector $g \neq 0$ (called admissible) such that $\int_{\mathcal{G}} |V_g f(x)|^2 d\mu(x) < \infty$ for all $f \in \mathcal{H}$. It is well-known [5] that in this case there exists a unique positive, self-adjoint, densely defined operator K on $\mathcal{H}$ whose domain $\mathcal{D}(K)$ consists of the admissible vectors and such that the orthogonality relation holds

$$\int_{\mathcal{G}} V_{g_1} f_1(x)\overline{V_{g_2} f_2(x)} dx \;=\; \langle K g_2, K g_1\rangle\langle f_1, f_2\rangle \tag{3.6}$$

for all $f_i \in \mathcal{H}$, $g_i \in \mathcal{D}(K)$, $i = 1, 2$. For unimodular groups K is a scalar multiple of the identity. It is an easy exercise to check that as a consequence of (3.6) we have the reproducing formula

$$V_g f \;=\; V_g f * V_g g \tag{3.7}$$

if g is normalized such that $\|Kg\| = 1$. The mapping $V_g : f \mapsto V_g f$ from $\mathcal{H}$ into $L^2(\mathcal{G})$ is isometric. We denote $\mathcal{D}_{\mathcal{A}}(K) := \mathcal{H}_{\mathcal{A}} \cap \mathcal{D}(K)$.

Lemma 3.5. *(a) The operator K commutes with the action of $\mathcal{A}$, i.e. $\sigma(A)K = K\sigma(A)$ for all $A \in \mathcal{A}$. Hence, K maps $D_{\mathcal{A}}(K)$ into $\mathcal{H}_{\mathcal{A}}$.*

(b) The space $\mathcal{D}_{\mathcal{A}}(K)$ is dense in $\mathcal{H}_{\mathcal{A}}$. Hence, if $\mathcal{H}_{\mathcal{A}}$ is non-trivial then also $\mathcal{D}_{\mathcal{A}}(K)$ is non-trivial.

Proof: (a) We know from [5] that $K = S^{-1/2}$ for some self-adjoint, positive, densely defined operator S that satisfies $\pi(x)S\pi(x)^{-1} \;=\; \Delta(x)^{-1}S$. Replacing x by $A(x)$ with $A \in \mathcal{A}$ and using (3.1) yields

$$\sigma(A)\pi(x)\sigma(A)^{-1}S\sigma(A)\pi(x)^{-1}\sigma(A)^{-1} \;=\; \Delta(A(x))^{-1}S.$$

Using the invariance of the modular function Δ under $\mathcal{A}$ we obtain

$$\pi(x)\left(\sigma(A)^{-1}S\sigma(A)\right)\pi(x)^{-1} \;=\; \Delta(x)^{-1}\sigma(A)^{-1}S\sigma(A).$$

From the positiveness of S if follows that also $\sigma(A)^{-1}S\sigma(A)$ is a positive operator. By Lemma 1 in [5] there exists a number $\lambda(A) \geq 0$ such that $\sigma(A)^{-1}S\sigma(A) \;=\; \lambda(A)S$. This implies $K\sigma(A) = \lambda^{-1/2}(A)\sigma(A)K$ for all $A \in \mathcal{A}$. Using the theorem of Duflo and Moore and the invariance of the Haar-measure of $\mathcal{G}$ we obtain for all $f \in \mathcal{H}$ and $g \in \mathcal{D}(K)$ that

$$\begin{aligned} \|Kg\|^2 \, \|f\|^2 &= \int_{\mathcal{G}} |\langle f, \pi(x)g\rangle|^2 dx = \int_{\mathcal{G}} |\langle \sigma(A)f, \pi(A(x))\sigma(A)g\rangle|^2 dx \\ &= \int_{\mathcal{G}} |\langle \sigma(A)f, \pi(x)\sigma(A)g\rangle|^2 dx = \|K\sigma(A)g\|^2 \, \|\sigma(A)f\|^2 = \|\lambda^{-1/2}(A)\sigma(A)Kg\|^2 \, \|f\|^2 \\ &= |\lambda(A)|^{-1}\|Kg\|^2 \, \|f\|^2. \end{aligned}$$

Together with the positiveness of $\lambda(A)$ this implies $\lambda(A) = 1$ which proves (a).

(b) The weakly defined operator $P_{\mathcal{A}} := \int_{\mathcal{A}} \sigma(A) dA$ is the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_{\mathcal{A}}$. Since K commutes with $\sigma(A)$ for all $A \in \mathcal{A}$ and since (by self-adjointness) K coincides with its closure the domain $\mathcal{D}(K)$ is invariant under $\sigma(A)$ for all $A \in \mathcal{A}$ (see also [14, Theorem 1.5.1]). Hence, it holds $P_{\mathcal{A}}(\mathcal{D}(K)) \subset \mathcal{D}(K)$. This implies in particular that $\mathcal{D}_{\mathcal{A}}(K) = P_{\mathcal{A}}(\mathcal{D}(K))$. Since $\mathcal{D}(K)$ is dense in $\mathcal{H}$ its image $\mathcal{D}_{\mathcal{A}}(K)$ under the projection $P_{\mathcal{A}}$ is dense in $\mathcal{H}_{\mathcal{A}} = P_{\mathcal{A}}(\mathcal{H})$. ■

Let us now collect some further properties of the restriction $\tilde{V}_g$ of V_g to $\mathcal{H}_A$ in a theorem as follows.

Theorem 3.6. *Suppose $g \in \mathcal{D}_{\mathcal{A}}(K)$ with $\|Kg\| = 1$.*

(a) For $\gamma \in \mathcal{D}_{\mathcal{A}}(K)$ and $f, h \in \mathcal{H}_{\mathcal{A}}$ then

$$\langle \tilde{V}_g f, \tilde{V}_\gamma h\rangle_{L^2(\mathcal{K},m)} = \langle K\gamma, Kg\rangle_{\mathcal{H}_{\mathcal{A}}} \langle f, h\rangle_{\mathcal{H}_{\mathcal{A}}}. \tag{3.8}$$

In particular, $\tilde{V}_g$ is an isometry from $\mathcal{H}_{\mathcal{A}}$ onto $L^2(\mathcal{K}, m) = L^2_{\mathcal{A}}(\mathcal{G})$.

(b) For $f \in \mathcal{H}_{\mathcal{A}}$ we have the reproducing formula

$$\tilde{V}_g f = \tilde{V}_g f * \tilde{V}_g g. \tag{3.9}$$

(c) The adjoint operator of $\tilde{V}_g : \mathcal{H}_{\mathcal{A}} \to L^2(\mathcal{K}, m)$ is given by

$$\tilde{V}_g^* : L^2(\mathcal{K}, m) \to \mathcal{H}_{\mathcal{A}}, \quad \tilde{V}_g^* F = \int_{\mathcal{K}} F(x)\tilde{\pi}(x)g \, dm(x) \tag{3.10}$$

where the integral is understood in a weak sense.

(d) Suppose $\gamma \in \mathcal{D}_{\mathcal{A}}(K)$ with $\langle K\gamma, Kg\rangle_{\mathcal{H}_{\mathcal{A}}} = 1$ and $f \in \mathcal{H}_{\mathcal{A}}$. Then the following inversion formula holds weakly:

$$f = \tilde{V}_\gamma^* \tilde{V}_g f = \int_{\mathcal{K}} \tilde{V}_g f(x)\tilde{\pi}(x)\gamma \, dm(x). \tag{3.11}$$

Proof: (a) is an immediate consequence of the orthogonality relation (3.6) and Lemma 3.5 and the reproducing formula in (b) follows from (3.7). For (c) let QF denote the right hand side of (3.10). Then for $h \in \mathcal{H}_\mathcal{A}$

$$\langle QF, h\rangle_{\mathcal{H}_\mathcal{A}} \;=\; \int_\mathcal{K} F(x)\langle\tilde{\pi}(x)g, h\rangle dm(x) \;=\; \langle F, \tilde{V}_g h\rangle_{L^2(\mathcal{K},m)}.$$

Hence $Q = \tilde{V}_g^*$. For (e) let $h \in \mathcal{H}_\mathcal{A}$. Using (3.8) we obtain

$$\begin{aligned}\langle \tilde{V}_\gamma^* \tilde{V}_g f, h\rangle_{\mathcal{H}_\mathcal{A}} &= \int_\mathcal{K} \tilde{V}_g f(x)\langle\tilde{\pi}(x)\gamma, h\rangle dm(x) \;=\; \int_\mathcal{K} \tilde{V}_g f(x)\overline{\tilde{V}_\gamma h(x)} dm(x) \\ &= \langle K\gamma, Kg\rangle_{\mathcal{H}_\mathcal{A}} \langle f, h\rangle_{\mathcal{H}_\mathcal{A}} \;=\; \langle f, h\rangle_{\mathcal{H}_\mathcal{A}}.\end{aligned}$$

Since h is arbitrary the theorem is proved. ■

Looking at the reproducing formula (3.9) one should think of formula (3.3) for the convolution. We further remark that the inversion formula (3.11) states on the one hand that we can reconstruct an element from $\mathcal{H}_\mathcal{A}$ by values of $\tilde{V}_g f$. Of course, to this end the values of $\tilde{V}_g f$ do not have to be computed on the whole group $\mathcal{G}$ but only for a single element out of each orbit $\mathcal{A}x$. On the other hand formula (3.11) states that we can represent any $f \in \mathcal{H}_\mathcal{A}$ by a continuous superposition of elements $\tilde{\pi}(x)g$ which are all contained in $\mathcal{H}_\mathcal{A}$.

4 Examples

4.1 The similitude group and radial functions

Let us return to our motivating example and discuss it in more detail in the context of the previous section. We consider the similitude group of the Euclidean plane $\mathcal{G} = \mathbb{R}^d \rtimes (\mathbb{R}_+^* \times SO(d))$, where $\mathbb{R}_+^*$ denotes the multiplicative group of positive real numbers. We assume that $d \geq 2$. For the case $d = 1$ (which is not very interesting in our context) some modifications have to be done at some places. The similitude group has left Haar-measure

$$\int_\mathcal{G} f(x)d\mu(x) \;=\; \int_{SO(d)} \int_{\mathbb{R}^d} \int_{\mathbb{R}_+^*} f(x, b, A)\frac{db}{b^{d+1}} dx dA$$

and modular function $\Delta(x, b, A) \;=\; b^{-d}$. A unitary irreducible representation of $\mathcal{G}$ on $L^2(\mathbb{R}^d)$ is given by

$$\pi(x, b, R)f(t) \;=\; b^{-d/2} f\left(b^{-1}R^{-1}(t-x)\right) \;=\; T_x D_b U_R f(t),$$

where the notation of section 2 is used. The corresponding voice transform is the continuous wavelet transform (2.1). We already know that π is square-integrable and the domain of the operator K is exactly the space of those functions $\psi \in L^2(\mathbb{R}^d)$ that satisfy (2.2).

A compact subgroup of $\mathcal{G}$ is given by $\mathcal{A} := \{(0,1,A) \mid A \in SO(d)\} \cong SO(d)$, which acts on $\mathcal{G}$ as inner automorphism group, i.e. if $B \in \mathcal{A}, \zeta = (x,b,A) \in \mathcal{G}$ then

$$B(\zeta) \;=\; B\zeta B^{-1} \;=\; (0,1,B)(x,b,A)(0,1,B^{-1}) \;=\; (Bx, b, BAB^{-1}).$$

With $\sigma := \pi_{|\mathcal{A}}$ it obviously holds $\pi(B(\zeta)) = \pi(B\zeta B^{-1}) = \sigma(B)\pi(\zeta)\sigma(B)^{-1}$ which is condition (3.1). Clearly, $\sigma(B)f(t) = f(B^{-1}t)$ and the space $\mathcal{H}_\mathcal{A}$ obviously is the space of all radial square-integrable functions on $\mathbb{R}^d$ denoted by $L^2_{rad}(\mathbb{R}^d)$.

The space $\mathcal{K} = \mathcal{A}(\mathcal{G})$ is the collection of all orbits $\mathcal{A}\zeta = \{(Bx, b, BAB^{-1}) | B \in SO(d)\}$. The operator $\tilde{\pi}(\zeta), \zeta = (x,b,A)$ on $\mathcal{H}_\mathcal{A} = L^2_{rad}(\mathbb{R}^d)$ turns out to be

$$\begin{aligned}\tilde{\pi}(\zeta)f(t) \;&=\int_{SO(d)} \pi(Bx,b,BAB^{-1})f(t)dB \;=\; b^{-d/2}\int_{SO(d)} f(b^{-1}BA^{-1}B^{-1}(t-Bx))dB\\ &= b^{-d/2}\int_{SO(d)} f(b^{-1}(t-Bx))dB \;=\; \tau_x D_b f(t),\end{aligned}$$

where τ_x denotes the generalized translation (2.4) on $\mathbb{R}^d$. Hence, $\tilde{\pi}(\mathcal{A}(x,b,A))$ depends only on $|x|$ and on b and therefore we may always choose $A = I$, the identity matrix. In fact, the set $\{\epsilon_{\mathcal{A}(x,b,I)} \mid x \in \mathbb{R}^d, b \in \mathbb{R}^*_+\}$ generates a subhypergroup $\mathcal{K}'$ of $\mathcal{K}$, i.e. the generated measure algebra $M(\mathcal{K}')$ is a closed subalgebra of $M(\mathcal{K})$. In other words if $F \in C^b_\mathcal{A}(\mathcal{G})$ then

$$\epsilon_{\mathcal{A}(x,b,I)} * \epsilon_{\mathcal{A}(y,c,I)}(F) \;=\; \int_{SO(d)} F(Ax+y, bc, I)dA \;=\; \int_{SO(d)} \epsilon_{\mathcal{A}(Ax+y,bc,I)}(F)dA.$$

The representation $\tilde{\pi}$ of $M(\mathcal{K})$ restricted to $M(\mathcal{K}')$ generates the same algebra of operators on $L^2_{rad}(\mathbb{R}^d)$. Clearly, an orbit $\mathcal{A}(x,b,I) = \{(Bx,b,I) \mid B \in SO(d)\}$ depends only on $|x|$ and $b \in \mathbb{R}^*_+$. Hence, it holds

$$\mathcal{K}' \;\cong\; \mathbb{R}_+ \times \mathbb{R}^*_+$$

and we may write $\tilde{\pi}(r,b) = \tilde{\pi}(\mathcal{A}(x,b,I))$ if $r = |x|$. The hypergroup $\mathcal{K}'$ can also be seen as a semidirect product of a Bessel-Kingman-Hypergroup and the group $\mathbb{R}^*_+$ as explained in [16].

For $g \in L^2_{rad}(\mathbb{R}^d)$ the restriction $\tilde{V}_\psi$ to $L^2_{rad}(\mathbb{R}^d)$ can be computed by formula (2.5). The projection m' of the Haar measure of $\mathcal{G}$ onto $\mathcal{K}'$ is given by

$$\int_{\mathcal{K}'} F(y)dm'(y) \;=\; \int_{\mathbb{R}_+}\int_{\mathbb{R}^*_+} F(r,b)\frac{db}{b^{d+1}}d\mu_d(r),$$

where $d\mu_d(r) = |S^{d-1}| r^{d-1} dr$.

Denoting $r = |x|, s = |y|$ and F_0 the function on $\mathbb{R}_+ \times \mathbb{R}_+^*$ corresponding to F such that $F(x,b) = F_0(|x|,b)$, the generalized translation on $\mathcal{K}'$ becomes

$$\begin{aligned}\mathcal{T}_{(y,c)}F(x,b) &= \int_{SO(d)} F(Ay+cx, cb)dA = \frac{1}{|S^{d-1}|}\int_{S^{d-1}} F(s\xi + cx, cb)dS(\xi) \\ &= \frac{|S^{d-2}|}{|S^{d-1}|}\int_0^\pi F_0(\sqrt{s^2 - 2rsc\cos\phi + c^2r^2}, cb)\sin^{d-2}(\phi)d\phi = \mathcal{T}_{(s,c)}F_0(r,b).\end{aligned}$$

From $\mathcal{A}(x,b,I)^{-1} = \mathcal{A}(-b^{-1}x, b^{-1}, I)$ follows that the left translation is given by

$$\mathcal{L}_{(r,b)} = \mathcal{T}_{(b^{-1}r, b^{-1})}, \quad r \in \mathbb{R}_+, b \in \mathbb{R}_+^*.$$

Theorems (3.3) and (3.6) immediately yield the following properties of $\tilde{V}_\psi$. Some of them were already noted in section 2.

Theorem 4.1. *Suppose $\psi \in L^2_{rad}(\mathbb{R}^d) \cap \mathcal{D}(K)$ with $\|K\psi\|_2^2 = c_\psi = 1$.*

(a) For $\gamma \in L^2_{rad}(\mathbb{R}^d) \cap \mathcal{D}(K)$ and $f, h \in L^2_{rad}(\mathbb{R}^d)$ then

$$\langle \tilde{V}_\psi f, \tilde{V}_\gamma h\rangle_{L^2(\mathcal{K}',m')} = \langle K\gamma, K\psi\rangle_{L^2(\mathbb{R}^d)}\langle f,h\rangle_{L^2(\mathbb{R}^d)}.$$

(b) The adjoint operator of $\tilde{V}_\psi$ is given by

$$\begin{aligned}&\tilde{V}_\psi^* : L^2(\mathcal{K}', m') \to L^2_{rad}(\mathbb{R}^d) \\ &\tilde{V}_\psi^* F(t) = \int_{\mathcal{K}'} F(x)\tilde{\pi}(x)\psi(t)dm'(x) = \int_{\mathbb{R}_+}\int_{\mathbb{R}_+^*} F(r,b)\tau_r D_b\psi(t)\frac{db}{b^{d+1}}d\mu_d(r).\end{aligned}$$

(c) (Inversion) Suppose $\gamma \in L^2_{rad}(\mathbb{R}^d)$ with $\langle K\gamma, K\psi\rangle = 1$ and $f \in L^2_{rad}(\mathbb{R}^d)$. Then

$$f(t) = \tilde{V}_\gamma^*\tilde{V}_\psi f(t) = \int_{\mathbb{R}_+}\int_{\mathbb{R}_+^*} \tilde{V}_\psi f(r,b)\tau_r D_b\gamma(t)\frac{db}{b^{d+1}}d\mu_d(r) \quad a.e.. \tag{4.1}$$

(d) (Covariance property) If $f \in L^2_{rad}(\mathbb{R}^d)$ and $r \in \mathbb{R}_+, b \in \mathbb{R}_+^*$ then

$$\tilde{V}_\psi(\tilde{\pi}(r,b)f) = \mathcal{L}_{(r,b)}(\tilde{V}_\psi f).$$

Observe that setting $\gamma = \psi$ in (4.1) yields (2.6).

4.2 The Heisenberg group and radial functions

Our second example is connected to the STFT. The (reduced) Heisenberg group $\mathbb{H}_d$ is the locally compact topological space $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{T}$ with group law

$$(x, \omega, \tau)(x', \omega', \tau') \ = \ (x + x', \omega + \omega', \tau\tau' e^{\pi i (x' \cdot \omega - x \cdot \omega')}).$$

It is unimodular and has Haar measure

$$\int_{\mathbb{H}_d} F(h) dh = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^1 F(x, \omega, e^{2\pi i t}) dt d\omega dx.$$

For $x, \omega \in \mathbb{R}^d$ define the translation $T_x f(t) := f(t - x)$ and the modulation $M_\omega f(t) := e^{2\pi i \omega \cdot t} f(t)$ for a function f on $\mathbb{R}^d$. The Schrödinger representation ρ is an irreducible unitary representation of the Heisenberg group acting on $L^2(\mathbb{R}^d)$ by

$$\rho(x, \omega, \tau) \ := \ \tau e^{\pi i x \cdot \omega} T_x M_\omega \ = \ \tau e^{-\pi i x \cdot \omega} M_\omega T_x.$$

The corresponding voice transform is essentially the short time Fourier transform, i.e.

$$\begin{aligned} V_g f(x, \omega, \tau) \ &= \ \langle f, \rho(x, \omega, \tau) g \rangle_{L^2(\mathbb{R}^d)} \ = \ \overline{\tau} \int_{\mathbb{R}^d} f(t) \overline{e^{-\pi i x \cdot \omega} M_\omega T_x g(t)} dt \\ &= \ \overline{\tau} e^{\pi i x \cdot \omega} \int_{\mathbb{R}^d} f(t) \overline{g(t - x)} e^{-2\pi i t \cdot \omega} dt \ = \ \overline{\tau} e^{\pi i x \cdot \omega} \, \mathrm{STFT}_g \, f(x, \omega). \end{aligned}$$

It is not difficult to see that the Schrödinger representation is square-integrable [11, Theorem 3.2.1] and the operator K is the identity, thus $\mathcal{D}(K) = \mathcal{H} = L^2(\mathbb{R}^d)$.

The automorphisms of $\mathbb{R}^d \times \mathbb{R}^d$ that extend to automorphisms of the Heisenberg group $\mathbb{H}_d$ are the elements of the symplectic group $Sp(d)$, which is defined as the subgroup of $GL(2d, \mathbb{R})$ leaving invariant the symplectic form

$$[(x, \omega), (x', \omega')] \ := \ x' \cdot \omega - x \cdot \omega'.$$

For more details on the Heisenberg group and its relation to the symplectic group the reader is referred to Gröchenig's excellent book [11, chapter 9].

A compact subgroup of $Sp(d)$ is given by

$$\mathcal{A} := \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \mid A \in SO(d) \right\} \cong SO(d).$$

An element $A \in SO(d) \cong \mathcal{A}$ acts on $\mathbb{H}_d$ by $A(x, \omega, \tau) = (Ax, A\omega, \tau)$. In the sequel we assume $d \geq 2$. In the case $d = 1$ (which is not a very illustrative example) one has to replace $SO(1) = 1$ by $O(1) = \{\pm 1\} \cong Z_2$ and put some adjustments where necessary.

As in the previous example we choose the natural representation σ of $SO(d)$ on $L^2(\mathbb{R}^d)$ given by $\sigma(A)f(t) = f(A^{-1}t)$ for $A \in SO(d), t \in \mathbb{R}^d$. Using the orthogonality of $A \in SO(d)$ we obtain

$$\begin{aligned}\rho(Ax, A\omega, \tau)\sigma(A)f(t) &= \tau e^{-\pi i(Ax\cdot A\omega)} e^{2\pi i A\omega\cdot t} f(A^{-1}(t - Ax)) \\ &= \tau e^{-\pi i(x\cdot\omega)} e^{2\pi i\omega\cdot A^{-1}t} f(A^{-1}t - x) \; = \; \sigma(A)\rho(x, \omega, \tau)f(t).\end{aligned}$$

Condition (3.1) is hence satisfied. As in the previous example we have $\mathcal{H}_{\mathcal{A}} = L^2_{rad}(\mathbb{R}^d))$.

The action of $\tilde{\rho}(x, \omega, \tau) = \rho(\epsilon_{\mathcal{A}(x,\omega,\tau)})$ on $L^2(\mathbb{R}^d)$ is given by

$$\tilde{\rho}(x, \omega, \tau)f(t) \; = \; \int_{SO(d)} \rho(Ax, A\omega, \tau)f(t)dA \; = \; \tau e^{\pi i x\cdot\omega} \int_{SO(d)} e^{2\pi i A\omega\cdot t} f(t - Ax)dA.$$

We already know from the general theory in section 3 that $\tilde{\rho}(x, \omega, \tau)$ maps $L^2_{rad}(\mathbb{R}^d)$ onto $L^2_{rad}(\mathbb{R}^d)$.

Lemma 4.2. *(a) Let $(x, \omega, \tau), (x', \omega', \tau') \in \mathbb{H}_d$. Both elements are contained in the same orbit under $\mathcal{A}$ if and only if $\tau = \tau', |x| = |x'|, |\omega| = |\omega'|$ and $\langle x, \omega\rangle = \langle x', \omega'\rangle$ where $|\cdot|$ and $\langle\cdot,\cdot\rangle$ denote the Euclidean norm and scalar product, respectively. Hence with*

$$\mathcal{S} := \mathbb{R}_+ \times \mathbb{R}_+ \times [-1, 1] \times \mathbb{T}$$

the orbit space $\mathcal{A}(\mathbb{H}_d)$ is parametrized by

$$\mathcal{K} := \mathcal{S} \setminus \{(r, s, t, \tau) \in \mathcal{S}, t \neq 1 \text{ and } (r = 0 \text{ or } s = 0)\} \tag{4.2}$$

(b) Consequently, $\tilde{\rho}(x, \omega, \tau)$ depends only on $|x|, |\omega|, \langle x, \omega\rangle$ and τ. If $f \in L^2_{rad}(\mathbb{R}^d)$ with corresponding function f_0 on $\mathbb{R}_+$ then the following formula applies

$$\begin{aligned}\tilde{\rho}(x, \omega, \tau)f(t) \;&=\; \tau e^{\pi i x\cdot\omega} \frac{|S^{d-2}|}{|S^{d-1}|} \times \\ \times &\int_0^\pi f_0(\sqrt{\theta^2 - 2r\theta\cos\phi + r^2}) e^{2\pi i\theta s\cos\alpha\cos\phi} \mathcal{B}_{d-1}(\theta s\sin\alpha\sin\phi)\sin^{d-2}\phi d\phi \\ =: &\;\tilde{\rho}(r, s, \cos\alpha, \tau)f_0(\theta),\end{aligned}$$

where $r = |x|, s = |\omega|, \langle x, \omega\rangle = rs\cos\alpha, \theta = |t|$ and $\mathcal{B}_d$ is the spherical Bessel function, i.e.

$$\mathcal{B}_d(t) \;=\; \frac{1}{|S^{d-1}|} \int_{S^{d-1}} e^{2\pi i t\langle\eta,\xi\rangle} dS(\xi), \quad \eta \in S^{d-1} \tag{4.3}$$

(independent of the choice of $\eta \in S^{d-1}$).

Proof: (a) Both elements are contained in the same orbit under $\mathcal{A}$ if and only if there exists a matrix $A \in SO(d)$ such that $(Ax, A\omega, \tau) = (x', \omega', \tau')$. Assume first that both elements are in the same orbit. Then it holds necessarily $\tau = \tau', |x| = |x'|$ and $|\omega| = |\omega'|$ implying that there exist elements $B, C \in SO(d)$ (not unique) such that $x' = Bx$ and $\omega' = C\omega$. Hence, we end up with the equations $A^{-1}Bx = x$ and $A^{-1}C\omega = \omega$ meaning that $D := A^{-1}B \in I(x)$ and $E := A^{-1}C \in I(\omega)$ where $I(x) := \{R \in SO(d), Rx = x\}$ denotes the isotropy subgroup of $x \in \mathbb{R}^d$ (which is isomorphic to $SO(d-1)$). Using the orthogonality of A we finally obtain

$$\langle x', \omega' \rangle = \langle Bx, C\omega \rangle = \langle ADx, AE\omega \rangle = \langle Dx, E\omega \rangle = \langle x, \omega \rangle.$$

Now assume that $\tau = \tau', |x| = |x'| \neq 0$ and $|\omega| = |\omega'| \neq 0$ and $\langle x', \omega' \rangle = \langle x, \omega \rangle$. Without loss of generality we may assume that $x = x' \in S^{d-1}$ and $\omega, \omega' \in S^{d-1}$. Since the sphere S^{d-1} is a two-point homogeneous space whose metric is given by $\cos d(x, y) = \langle x, y \rangle$ there exists a matrix $A \in I(x)$ such that $A\omega = \omega'$ implying $(Ax, A\omega, \tau) = (x', \omega', \tau')$. The cases $x = 0$ or $\omega = 0$ are trivial.

In order to have an explicit correspondence between $\mathcal{A}(\mathbb{H}_d)$ and the set in (4.2) fix elememts $\eta, \xi \in S^{d-1}$ with $\langle \eta, \xi \rangle = 0$. For an element $(r, s, t, \tau) \in \mathcal{K}$ put $(x, \omega, \tau) := (r\eta, s(\cos(\alpha)\eta + \sin(\alpha)\xi), \tau) \in \mathbb{H}_d$ where $\cos \alpha = t$.

The first assertion of (b) is an immediate consequence of (a) which can also be easily verified directly. For the proof of the second assertion of (b) we use the following rule for integration on the sphere. If $\xi = \cos\phi\eta + \sin\phi\xi' \in S^{d-1}$ where $\eta \in S^{d-1}$ is fixed and $\xi' \in S^{d-1}$ with $\langle \eta, \xi' \rangle = 0$ (obviously the set of those ξ' is isomorphic to S^{d-2}) then

$$\int_{S^{d-1}} g(\xi) dS(\xi) \;=\; \int_0^\pi \int_{S^{d-2}} g(\cos\phi\,\eta + \sin\phi\,\xi') dS^{d-2}(\xi') \sin^{d-2}\phi d\phi.$$

Let ω' such that $\omega = s(\cos\alpha\,\eta + \sin\alpha\,\omega')$ with $\langle \eta, \omega' \rangle = 0$. Using Weil's formula [9, Theorem 2.49] for the Haar measure on $SO(d)$ we obtain

$$\begin{aligned}
&\int_{SO(d)} e^{2\pi i A\omega\cdot t} f(t - Ax) dA = \int_{SO(d)} e^{2\pi i \omega\cdot A^{-1}t} f(A^{-1}t - x) dA \\
&= \frac{1}{|S^{d-1}|} \int_{S^{d-1}} e^{2\pi i \theta\omega\cdot\xi} f(\theta\xi - x) dS(\xi) \\
&= \frac{1}{|S^{d-1}|} \int_0^\pi \int_{S^{d-2}} e^{2\pi i\theta s(\cos\alpha\,\eta + \sin\alpha\,\omega')\cdot(\cos\phi\,\eta + \sin\phi\,\xi')} \times \\
&\qquad \times\; f(\theta(\cos\phi\,\eta + \sin\phi\,\xi') - r\eta) dS^{d-2}(\xi') \sin^{d-2}\phi d\phi \\
&= \frac{1}{|S^{d-1}|} \int_0^\pi f_0(\sqrt{\theta^2 - 2r\theta\cos\phi + r^2}) e^{2\pi i\theta s\cos\alpha\cos\phi} \times \\
&\qquad \times \int_{S^{d-2}} e^{2\pi i\theta s\sin\alpha\sin\phi\,\omega'\cdot\xi'} dS^{d-2}(\xi') \sin^{d-2}\phi d\phi
\end{aligned}$$

$$
\begin{aligned}
&= \frac{|S^{d-2}|}{|S^{d-1}|}\int_0^\pi f_0(\sqrt{\theta^2-2r\theta\cos\phi+r^2})e^{2\pi i\theta s\cos\alpha\cos\phi}\mathcal{B}_{d-1}(\theta s\sin\alpha\sin\phi)\sin^{d-2}\phi d\phi \\
&=: \Omega(r,s,\cos\alpha)f_0(\theta). \qquad (4.4)
\end{aligned}
$$

This finishes the proof. ∎

We obtain an easy corollary which nevertheless seems not to be present in the literature.

Corollary 4.3. *Let $f,g\in L^2_{rad}(\mathbb{R}^d)$. Then the short time Fourier transform $\mathrm{STFT}_g f(x,\omega)$ depends only on $|x|,|\omega|$ and $\langle x,\omega\rangle$.*

Note that the spherical Bessel function (4.3) can be expressed by means of the Bessel function J_α of the first kind

$$
\mathcal{B}_d(t) \;=\; \Gamma(\alpha+1)(\pi t)^{-\alpha}J_\alpha(2\pi t), \quad \alpha=\frac{d-2}{2}.
$$

The operator $\Omega(r,s,\cos\alpha)$ may be viewed as a generalized combined translation and modulation For special values it simplifies a little,

$$
\begin{aligned}
\Omega(0,s,1)f_0(\theta) &= f_0(\theta)\mathcal{B}_d(s\theta),\\
\Omega(r,0,1)f_0(\theta) &= \frac{|S^{d-2}|}{|S^{d-1}|}\int_0^\pi f_0(\sqrt{\theta^2-2r\theta\cos\phi+r^2})\sin^{d-2}\phi d\phi \;=\; \tau_r f_0(\theta).
\end{aligned}
$$

With $g\in L^2_{rad}(\mathbb{R}^d)$ let us consider now the restriction $\tilde V_g$ of V_g to $L^2_{rad}(\mathbb{R}^d)$. Interpreting it as a function on $\mathcal{K}$ it holds

$$
\begin{aligned}
V_g(x,\omega,\tau) &= \tilde V_g f(r,s,\cos\alpha,\tau) \;=\; \int_{\mathbb{R}^d} f(t)\overline{\tilde\rho(r,s,\cos\alpha,\tau)g(t)}dt\\
&= \overline{\tau} e^{-\pi i rs\cos\alpha}\int_0^\infty f_0(\theta)\overline{\Omega(r,s,\cos\alpha)g_0(\theta)}d\mu_d(\theta), \qquad (4.5)
\end{aligned}
$$

where $d\mu_d(\theta)=|S^{d-1}|\theta^{d-1}d\theta$ and $(x,\omega,\tau)=(r\eta,s(\cos(\alpha)\eta+\sin(\alpha)\xi),\tau)$ with fixed $\tau,\eta\in S^{d-1}$ such that $\langle\eta,\xi\rangle=0$. The integral is the STFT of a radial function,

$$
\mathrm{STFT}_g f(x,\omega) \;=\; \int_0^\infty f_0(\theta)\overline{\Omega(r,s,\cos\alpha)g_0(\theta)}d\mu_d(\theta) \;=:\; \widetilde{\mathrm{STFT}}_{g_0}f_0(r,s,\cos\alpha) \qquad (4.6)
$$

where $r=|x|, s=|\omega|, rs\cos\alpha=\langle x,\omega\rangle$.

Remark 4.1. *As already mentioned in the introduction there were recently introduced other transforms on $\mathbb{R}_+$ (or even on more general hypergroups) which were claimed to be analogues of the STFT on $\mathbb{R}^d$ [3, 4]. These approaches try to mimic (in two different* *ways) the STFT directly by using the generalized translation (2.4) of the Bessel-Kingman hypergroup (a hypergroup on $\mathbb{R}_+$) and the Hankel transform as a substitute for the Fourier transform. It appears however that these transforms lack some important properties which suggests that the transform (4.6) is more natural despite the fact that it depends on three parameters rather than the perhaps expected two parameters.*

By definition the generalized translation $\mathcal{T}$ on the hypergroup $\mathcal{A}(\mathbb{H}_d)$ is given by

$$\mathcal{T}_{(x,\omega,\tau)}F(x',\omega',\tau') \;=\; \int_{SO(d)} F(Ax+x', A\omega+\omega', \tau\tau' e^{\pi i(x'\cdot A\omega - Ax\cdot\omega')})dA,$$

where dA denotes the Haar measure on $SO(d)$ and $(x,\omega,\tau),(x',\omega',\tau') \in \mathbb{H}_d$. We associate to a function F on $\mathbb{H}_d$ invariant under $\mathcal{A}$ a function F_0 on $\mathcal{K}$ by

$$F_0(r,s,\cos\alpha,\tau) \;=\; F(r\eta, s(\cos(\alpha)\eta + \sin(\alpha)\xi), \tau), \quad (r,s,\cos(\alpha),\tau) \in \mathcal{K},$$

where η,ξ are as above. As before we use the same symbol for the generalized translation for invariant functions on $\mathbb{H}_d$ and for functions on $\mathcal{K}$. Let us compute $\mathcal{T}$ explicitely for the case $d=2$.

Lemma 4.4. *Let $d=2$ and $F \in C^b_{\mathcal{A}}(\mathbb{H}_2)$ with associated function F_0 on $\mathcal{K}$. Further let $(r,s,\cos\alpha,\tau),(r',s',\cos\alpha',\tau') \in \mathcal{K}$. Then it holds*

$$\begin{aligned}
&\mathcal{T}_{(r,s,\cos\alpha,\tau)}F_0(r',s',\cos\alpha',\tau') \\
&= \int_0^{2\pi} F_0\Big(\sqrt{r^2+2rr'\cos\theta+(r')^2}, \sqrt{s^2+2ss'\cos(\theta+\alpha'-\alpha)+(s')^2}, \\
&\qquad \frac{rs\cos\alpha + r's'\cos\alpha' + r's\cos(\alpha-\theta) + rs'\cos(\alpha'+\theta)}{\sqrt{r^2+2rr'\cos\theta+(r')^2}\sqrt{s^2+2ss'\cos(\theta+\alpha'-\alpha)+(s')^2}}, \\
&\qquad \tau\tau' e^{\pi i(r's\sin(\theta+\alpha) - rs'\sin(\alpha'+\theta))}\Big)d\theta. \qquad (4.7)
\end{aligned}$$

Proof: We choose $\eta = e_1 := (1,0)^T$ and $\xi = e_2 := (0,1)^T$. The group $SO(2)$ is parametrized by $[0,2\pi)$ by means of the matrices

$$A_\theta \;:=\; \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

and has Haar measure $\int_{SO(2)} G(A)dA = \int_0^{2\pi} G(A_\theta)d\theta$. After some calculations involving trigonometric identities we obtain

$$\int_{SO(2)} F(Ax+x', A\omega+\omega', \tau\tau' e^{\pi i(x'\cdot A\omega - Ax\cdot\omega')})dA$$
$$= \int_0^{2\pi} F\left(\left(\begin{array}{c} r\cos\theta + r' \\ -r\sin\theta \end{array}\right), \left(\begin{array}{c} s\cos(\alpha-\theta)+s'\cos\alpha' \\ s\sin(\alpha-\theta)+s'\sin\alpha' \end{array}\right),\right. \tag{4.8}$$
$$\left. \tau\tau' e^{\pi i(r's\sin(\alpha+\theta) - rs'\sin(\alpha'+\theta))}\right) d\theta,$$

where as before $(x,\omega,\tau) = (re_1, s(\cos\alpha e_1 + \sin\alpha e_2), \tau)$ and $(x',\omega',\tau') = (r'e_1, s'(\cos\alpha' e_1 + \sin\alpha' e_2), \tau')$. Noting that $F(x,\omega,\tau) = F_0(|x|, |\omega|, \frac{\langle x,\omega\rangle}{|x|\,|\omega|}, \tau)$ and computing the Euclidean norms of the first and second entry of F in (4.8) and their scalar product gives (4.7). ■

Since $(x,\omega,\tau)^{-1} = (-x,-\omega,\overline{\tau})$ the involution of the hypergroup $\mathcal{K}$ is given by $(r,s,\cos\alpha,\tau)^{\tilde{}} = (r,s,\cos\alpha,\overline{\tau})$ and hence

$$\mathcal{L}_{(r,s,\cos\alpha,\tau)} = \mathcal{T}_{(r,s,\cos\alpha,\overline{\tau})}.$$

A straightforward calculation shows that the projection m of the Haar measure of $\mathbb{H}_d$ onto $\mathcal{K}$ is given by

$$\int_{\mathcal{K}} F(x)dm(x)$$
$$= \frac{|S^{d-2}|}{|S^{d-1}|}\int_0^{2\pi}\int_0^{\infty}\int_0^{\infty}\int_0^{\pi} F(r,s,\cos\alpha, e^{2\pi i t})\sin^{d-2}(\alpha)d\alpha d\mu_d(r)d\mu_d(s)dt.$$

Now we are ready to apply Theorems 3.3 and 3.6 to our situation.

Theorem 4.5. *Let $g \in L^2_{rad}(\mathbb{R}^d)$ such that $\|g\|_{L^2(\mathbb{R}^d)} = 1$.*

(a) For $\gamma, f, h \in L^2_{rad}(\mathbb{R}^d)$ then

$$\langle \tilde{V}_g f, \tilde{V}_\gamma h\rangle_{L^2(\mathcal{K})} = \langle \gamma, g\rangle_{L^2(\mathbb{R}^d)} \langle f, g\rangle_{L^2(\mathbb{R}^d)}$$

In particular, $\tilde{V}_g$ is an isometry from $L^2_{rad}(\mathbb{R}^d)$ onto $L^2(\mathcal{K})$.

(b) The adjoint operator of $\tilde{V}_g$ is given by

$$\tilde{V}_g^* : L^2(\mathcal{K}) \to L^2_{rad}(\mathbb{R}^d),$$
$$\tilde{V}_g^* F(t) = \int_{\mathcal{K}} F(r,s,\cos\alpha,\tau)\tilde{\rho}(r,s,\cos\alpha,\tau)g(t)dm(r,s,\alpha,\tau).$$

(c) Suppose $\gamma \in L^2_{rad}(\mathbb{R}^d)$ with $\langle \gamma, g\rangle = 1$. Then the following inversion formula holds

$$\begin{aligned} f(t) &= \tilde{V}_\gamma^* \tilde{V}_g f(t) \\ &= \int_{\mathcal{K}} \tilde{V}_g f(r,s,\cos\alpha,\tau)\tilde{\rho}(r,s,\cos\alpha,\tau)\gamma(t) dm(r,s,\alpha,\tau) \quad a.e.. \end{aligned}$$

(d) (Covariance property) Let $f \in L^2_{rad}(\mathbb{R}^d)$ and $(r,s,\cos\alpha,\tau) \in \mathcal{K}$. Then it holds

$$\tilde{V}_g(\tilde{\rho}(r,s,\cos\alpha,\tau)f) = \mathcal{L}_{(r,s,\cos\alpha,\tau)}\tilde{V}_g f.$$

Of course, these results immediately imply corresponding results for the STFT. We only state one property explicitely.

Corollary 4.6. *Suppose $g, \gamma \in L^2_{rad}(\mathbb{R}^d)$ with $\langle \gamma, g\rangle = 1$. Then $ST\tilde{F}T_g$ is inverted on $L^2_{rad}(\mathbb{R}^d)$ by the formula*

$$f(t) = \frac{|S^{d-2}|}{|S^{d-1}|}\int_0^\infty \int_0^\infty \int_0^\pi \mathrm{ST\tilde{F}T}_g f(r,s,\cos\alpha)\Omega(r,s,\cos\alpha)\gamma(t)\sin^{d-2}(\alpha)d\alpha d\mu_d(s)d\mu_d(r)$$

for almost all $t \in \mathbb{R}^d$.

Proof: It holds

$$\tilde{V}_g f(r,s,\cos\alpha,\tau)\tilde{\rho}(r,s,\cos\alpha,\tau)\gamma = \mathrm{ST\tilde{F}T}_g f(r,s,\cos\alpha)\Omega(r,s,\cos\alpha)\gamma,$$

the latter being independent of τ. Thus the assertion follows from the previous Theorem. ∎

With this example we have in some sense settled the foundations for radial time-frequency analysis. There are many open questions left such as investigating radial Gabor frames. Having the formula (4.4) for the generalized combined translation and modulation Ω in mind it will probably be very involved to get results in this direction. However, using the abstract approach and a generalization of coorbit space theory [7, 8, 10] the author was already successful in proving some results concerning invariant frames which will appear in a future paper. Nevertheless, it is rather unlikely that all problems concerning radial Gabor frames can be attacked by abstract methods.

4.3 Reflection symmetries

Let us consider both previous examples where the symmetry group $\mathcal{A} = SO(d)$ is replaced by a finite reflection group W, which is a finite subgroup of $O(d)$. Sometimes W is called Coxeter group or (in the context of Lie algebras) Weyl group [12], [6, chapter 4]. Of course,

W acts in the same way as $SO(d)$ on the similitude group and the Heisenberg group. The representation σ is again the natural action on $L^2(\mathbb{R}^d)$. The Haar measure of W is given by a sum, $\int_W f(w)d\mu(w) = |W|^{-1} \sum_{w \in W} f(w)$.

A Coxeter group can be described using a root system. For $0 \neq u \in \mathbb{R}^d$ denote by R_u the reflection at the hyperplane perpendicular to u, i.e.

$$R_u x = x - 2\frac{\langle x, u\rangle}{|u|^2} u, \quad x \in \mathbb{R}^d.$$

A root system is a finite set $S \subset \mathbb{R}^d \setminus \{0\}$ such that $R_u v \in S$ for all $u, v \in S$. The Coxeter group $W = W(S)$ associated to the root system S is the (finite!) subgroup of $O(d)$ generated by the reflections $\{R_u, u \in S\}$. For a classification of all Coxeter groups see [12]. The complement of the union of the hyperplanes $\bigcup_{u \in S} u^\perp$ splits into several open connected components, called Weyl chambers. For an arbitrary *closed* Weyl chamber C it holds $\bigcup_{w \in W} w(C) = \mathbb{R}^d$. Hence, a function on $\mathbb{R}^d$ invariant under $W(S)$ is determined by its values on a closed Weyl chamber C and can hence be regarded as a function on C.

Applying the results of section 3 to this setting yields wavelet analysis and time-frequency analysis on Weyl chambers. For reasons of length we skip the details at this place.

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Holger Rauhut
Zentrum Mathematik, GK-MA
Technische Universität München
Boltzmannstr. 3, D–85747 Garching
Germany
rauhut@ma.tum.de