Generalized hypergroups and orthogonal polynomials

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Abstract

The concept of semi-bounded generalized hypergroups (SBG hypergroups) is developed which are more special then generalized hypergroups introduced by Obata and Wildberger and which are more general then discrete hypergroups or even discrete signed hypergroups. The convolution of measures and functions is studied. In case of commutativity we define the dual objects and prove some basic theorems of Fourier analysis. Furthermore, we investigate the relationship between orthogonal polynomials and generalized hypergroups. We discuss the Jacobi polynomials as an example.
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1 Introduction

Locally compact hypergroups were independently introduced around the 1970's by Dunkl [4], Jewett [7] and Spector [18]. They generalize the concepts of locally compact groups with the purpose of doing standard harmonic analysis. Similar structures had been studied earlier in the 1950's by Berezansky and colleagues, and even earlier in works of Delsarte and Levitan.

Later on results of harmonic analysis on hypergroups were transferred to different applications. For example a Bochner theorem is used essentially in the context of weakly stationary processes indexed by hypergroups, see [10] and [12]. Hypergroup structure is also heavily used in probability theory, see the monograph [2], and in approximation with respect to orthogonal polynomial sequences, see [5] and [11]. However, not the whole set of axioms (see [2]) is used in these application areas. So concentrating on orthogonal polynomials, Obata and Wildberger studied in [13] a very general concept and called it “generalized hypergroups”. The purpose of the present paper is to derive results of harmonic analysis for generalized hypergroups in more detail than in [13]. Our main interest is to include all orthogonal polynomial systems with respect to a compactly supported orthogonalization measure in our investigations.

2 Semi-bounded generalized hypergroups

The discrete structure of a generalized hypergroup was introduced by Obata and Wildberger in [13]. Let us recall the basic definition.

Definition 2.1 *A generalized hypergroup is a pair* $(\mathcal{K}, \mathcal{A}_0)$, *where* $\mathcal{A}_0$ *is a* $*$*-algebra over* $\mathbb{C}$ *with unit* c_0 *and* $\mathcal{K} = \{c_k, k \in K\}$ *is a countable subset of* $\mathcal{A}_0$ *containing* c_0 *that satisfies the following axioms.*

(A1) $\mathcal{K}^* \;=\; \mathcal{K}$.

(A2) $\mathcal{K}$ *is a linear basis of* $\mathcal{A}_0$, *i.e., every* $a \in \mathcal{A}_0$ *admits a unique expression of the form* $a = \sum_n \alpha_n c_n$ *with only finitely many nonzero* $\alpha_i \in \mathbb{C}$.

(A3) The structure constants or linearization coefficients $g(n,m,k) \in \mathbb{C}$ *which are defined by*

$$c_n c_m \;=\; \sum_k g(n,m,k) c_k$$

satisfy the condition

$$g(n,m,0) \quad \begin{cases} > \; 0 & \text{if } c_n^* = c_m, \\ = \; 0 & \text{if } c_n^* \neq c_m. \end{cases}$$

A generalized hypergroup is called hermitian if $c_n^* = c_n$, *commutative if* $c_n c_m = c_m c_n$, *real if* $g(n,m,k) \in \mathbb{R}$, *positive if* $g(n,m,k) \geq 0$ *and normalized if* $\sum_j g(n,m,j) = 1$ *for all* n,m,k.

A bijection $\tilde{}$ on K is defined by

$$c_{\tilde{n}} = c_n^*. \tag{1}$$

Further, let

$$h(n) \;=\; g(\tilde{n}, n, 0)^{-1}. \tag{2}$$

Due to $(A3)$ we have $h(n) > 0$ for all n and $h(0) = 1$. If $\mathcal{K}$ is hermitian or commutative then $h(n) = h(\tilde{n})$. In the following lemma some useful properties of the structure constants are summarized.

Lemma 2.2 *The structure constants fulfill the following equalities*

$$g(n,0,k) \;=\; g(0,n,k) \;=\; \delta_{nk}, \tag{3}$$

$$g(n,m,k) \;=\; \overline{g(\tilde{m},\tilde{n},\tilde{k})}, \tag{4}$$

$$h(m) g(n,m,k) \;=\; h(k) g(\tilde{k}, n, \tilde{m}) \quad \text{and} \tag{5}$$

$$\sum_k g(n,m,k) g(k,l,j) \;=\; \sum_k g(n,k,j) g(m,l,k) \quad \text{for all } n,m,l,j. \tag{6}$$

PROOF: For (3)–(5) see [13, Lemma 1.1]. Now, on the one hand we have $(c_n c_m)c_l = \sum_{k,j} g(n,m,k)g(k,l,j)c_j$ and on the other hand $c_n(c_m c_l) = \sum_{k,j} g(m,l,k)g(n,k,j)c_j$. From the associativity of $\mathcal{A}_0$ and from the linear independence of the set $\mathcal{K}$ follows (6). ■

We define translation operators L_n, $\overline{L_n}$ for complex valued functions f on K by

$$L_n f(m) \;=\; \sum_k g(n,m,k)f(k) \quad \text{and} \quad \overline{L_n} f(m) \;=\; \sum_k \overline{g(n,m,k)}f(k).$$

Given f the function $\tilde{f}$ is defined by $\tilde{f}(n) = f(\tilde{n})$.

Lemma 2.3 *For f, g with finite support and all $n \in K$ it holds that*

$$\sum_m L_n f(m)g(m)h(m) \;=\; \sum_m f(m)L_{\tilde{m}}\tilde{g}(n)h(m) \;=\; \sum_m f(m)(\overline{L_{\tilde{n}}}g)(m)h(m). \tag{7}$$

PROOF: We use (5) and (4) to obtain

$$\begin{aligned}\sum_m (L_n f)(m)g(m)h(m) \;&=\; \sum_{m,k} g(\tilde{k},n,\tilde{m})h(k)f(k)g(m) \;=\; \sum_k f(k)L_{\tilde{k}}\tilde{g}(n)h(k)\\ &=\; \sum_{k,m} f(k)\overline{g(\tilde{n},k,m)}g(m)h(k) \;=\; \sum_k f(k)(\overline{L_{\tilde{n}}}g)(k)h(k).\end{aligned}$$ ■

We write $\nu(k) = \nu(\{k\})$ for a discrete measure ν on K. Let ϵ_n denote the Dirac-measure at $n \in K$, i.e., $\epsilon_n(k) = 1$ if $k = n$ and $\epsilon_n(k) = 0$ else.

Definition 2.4 *A positive discrete measure $\omega \neq 0$ on K is called (left) Haar measure if for all f with finite support and all $n \in K$ it holds*

$$\sum_m L_n f(m)\omega(m) \;=\; \sum_m f(m)\omega(m).$$

Theorem 2.5 *A Haar measure exists if and only if $\mathcal{K}$ is normalized. In that case all Haar measures ω are determined by $\omega = \alpha h$, $\alpha > 0$.*

PROOF: Let us assume that there exists a Haar measure ω. Due to (A3) we get

$$h(n)^{-1}\omega(n) \;=\; \sum_m g(\tilde{n},m,0)\omega(m) \;=\; \sum_m L_{\tilde{n}}\epsilon_0(m)\omega(m) \;=\; \sum_m \epsilon_0(m)\omega(m) \;=\; \omega(0),$$

which yields $\omega(n) = \omega(0)h(n)$. Now, let $\omega = \alpha h$. It suffices to consider $f = \epsilon_k$. By (5) we get

$$\sum_m L_n\epsilon_k(m)\omega(m) \;=\; \sum_m \omega(k)g(\tilde{k},n,\tilde{m}) = \sum_m \epsilon_k(m)\omega(m)\sum_m g(\tilde{k},n,m).$$

Hence, ω is a Haar measure if and only if $\mathcal{K}$ is normalized. ■

In order to develop their theory further Obata and Wildberger took care of the functional $\phi_0 : \mathcal{A}_0 \to \mathbb{C}$ defined by

$$\phi_0\left(\sum_n \alpha_n c_n\right) \;=\; \alpha_0,$$

and focused on the following property. A generalized hypergroup $(\mathcal{K}, \mathcal{A}_0)$ is said to satisfy property (B) if for all n there exists $\kappa(n) \geq 0$ such that

$$|\phi_0(b^* c_n b)| \;\leq\; \kappa(n)\phi_0(b^* b) \qquad \text{for all } b \in \mathcal{A}_0.$$

We focus on a stronger property than Obata and Wildberger.

Definition 2.6 *A generalized hypergroup $(\mathcal{K}, \mathcal{A}_0)$ is called a semi-bounded generalized hypergroup (SBG hypergroup) if, additionally, the following axiom is valid.*

(A4) For the structure constants it holds

$$\gamma(n) \;=\; \sup_m \sum_k |g(n,m,k)| < \infty \quad \textit{for all } n. \tag{8}$$

A generalized hypergroup is called bounded if it is semi-bounded and γ is bounded.

An SBG hypergroup is satisfying property (B) with $\kappa(n) = \gamma(n)$, see [13, Theorem 4.1]. It holds

$$\gamma(n) \;\geq\; \max(h(\tilde{n})^{-1}, 1).$$

By simple arguments we have

$$\gamma(\tilde{m}) \;=\; \sup_n \sum_k |g(n,m,k)|. \tag{9}$$

If $\mathcal{K}$ is hermitian or commutative then $\gamma(\tilde{n}) = \gamma(n)$, and if $\mathcal{K}$ is positive and normalized then $\gamma(n) = 1$ for all n.

3 Convolution of measures and functions

Clearly, both measures and functions on K can be identified with sequences indexed by K. However, we make a distinction anyway, since the natural definition of a convolution is different for measures and functions.

So for discrete complex measures μ, ν on K we define a convolution by

$$(\mu * \nu)(k) \;=\; \sum_{n,m} g(n,m,k)\mu(n)\nu(m) \tag{10}$$

whenever the sum on the right hand side is finite for all k. A short calculation shows $\epsilon_0 * \mu = \mu * \epsilon_0 = \mu$, i.e., ϵ_0 is the unit element for this convolution. For two Dirac measures we get $\epsilon_n * \epsilon_m \;=\; \sum_k g(n,m,k)\epsilon_k$, and $\operatorname{supp} \epsilon_n * \epsilon_m$ is finite.

In order to investigate the convergence of the sum in (10) we introduce the spaces

$$M(K) \;=\; \left\{\mu \text{ measure on } K,\ |\mu|(K) \;=\; \sum_n |\mu(n)| \;<\; \infty\right\},\ \|\mu\| \;=\; |\mu|(K),$$

$$M_\gamma(K) \;=\; \left\{\mu \in M(K),\ |\gamma\mu|(K) \;=\; \sum_n |\mu(n)|\gamma(n) \;<\; \infty\right\},\ \|\mu\|_\gamma \;=\; |\gamma\mu|(K).$$

The space $M_{\tilde{\gamma}}(K)$ and the norm $\|\mu\|_{\tilde{\gamma}}$ is defined analogously.

Lemma 3.1 *(i) If $\mu \in M_\gamma(K)$ and $\nu \in M(K)$ then $\mu * \nu \in M(K)$ and $\|\mu * \nu\| \leq \|\mu\|_\gamma \|\nu\|$.*

(ii) If $\mu \in M(K)$ and $\nu \in M_{\tilde{\gamma}}(K)$ then $\mu * \nu \in M(K)$ and $\|\mu * \nu\| \leq \|\mu\| \|\nu\|_{\tilde{\gamma}}$.

PROOF: (i) We have by Fubini

$$\|\mu * \nu\| \;\leq\; \sum_{n,m} \sum_k |g(n,m,k)||\mu(n)||\nu(m)| \;\leq\; \sum_{n,m} \gamma(n)|\mu(n)||\nu(m)| \;=\; \|\mu\|_\gamma \|\nu\|.$$

The proof of (ii) is analogous additionally using (9). ■

Lemma 3.2 *The convolution $*$ is associative, i.e., $(\mu * \nu) * \rho = \mu * (\nu * \rho)$ whenever both expressions exist in the sense of Lemma 3.1.*

PROOF: It suffices to proof the associativity for Dirac measures $(\epsilon_n * \epsilon_m) * \epsilon_l \;=\; \epsilon_n * (\epsilon_m * \epsilon_l)$ For that purpose use (6). ■

Lemma 3.3 *(i) It holds that $(\epsilon_n * \epsilon_m)^{\tilde{}} \;=\; \overline{\epsilon_{\tilde{m}} * \epsilon_{\tilde{n}}}$.*

(ii) It holds $0 \in \operatorname{supp} \epsilon_n * \epsilon_{\tilde{m}}$ if and only if $n = \tilde{m}$

(iii) If $\mathcal{K}$ is normalized then $\epsilon_n * \epsilon_m(K) = 1$ for all n, m.

(iv) It holds $\epsilon_n * \epsilon_{\tilde{n}}(0) = h(\tilde{n})^{-1} > 0$.

(v) It holds $\|\epsilon_n * \epsilon_m\| \;\leq\; \min(\gamma(n), \gamma(\tilde{m}))$.

PROOF: Using (5) we obtain (i), and application of axiom (A3) gives (ii). For (v) we have by definition $\|\epsilon_n * \epsilon_m\| \;=\; \sum_k |g(n,m,k)| \;\leq\; \gamma(n)$. The second inequality is achieved analogously by using (9). The assertions (iii) and (iv) are clear. ■

Now, we are able to compare the concept of an SBG hypergroup with that of a discrete hypergroup, see for example [5], or a discrete signed hypergroup, see [14]. Our previous results give the following theorem.

Theorem 3.4 *(i) If $\mathcal{K}$ is a real, normalized and bounded generalized hypergroup then its index set K with convolution $*$ as defined in (10) and involution $\tilde{}$ as defined in (1) is a discrete signed hypergroup.*

(ii) If $\mathcal{K}$ is a positive and normalized SBG hypergroup then its index set K with convolution $*$ and involution $\tilde{}$ is a discrete hypergroup.

(iii) Let $(K, \star, \tilde{})$ be a discrete signed hypergroup. Put $\mathcal{K} = \{\epsilon_k, k \in K\}$ and let $\mathcal{A}_0$ be the vector space of all finite linear combinations of Dirac measures $\epsilon_k \in \mathcal{K}$. Further, let $\star$ be the multiplication in $\mathcal{A}_0$ and put $\epsilon_k^* = \epsilon_{\tilde{k}}$ as involution on $\mathcal{K}$, which is linearly extended to $\mathcal{A}_0$. Then $(\mathcal{K}, \mathcal{A}_0)$ is a real, bounded and normalized generalized hypergroup.

(iv) If $(K, *, \tilde{\ })$ is a discrete hypergroup then the construction in (iii) yields a positive and normalized SBG hypergroup.

Next let us introduce the convolution of functions.

Definition 3.5 *Let f and g be functions on K with finite support. The convolution of those functions is defined by*

$$(f \star g)(m) \;=\; \sum_n f(n)(\overline{L_{\tilde{n}}}g)(m)h(n). \tag{11}$$

Lemma 3.6 *If f and g have finite support then $f \star g$ has finite support.*

PROOF: By definition $(f \star g)(m) = \sum_{n,k} f(n)\overline{g(\tilde{n}, m, k)}g(k)h(n)$. Hence, $\operatorname{supp} f \star g \subset \bigcup_{n \in \operatorname{supp} f, k \in \operatorname{supp} g} M_{\tilde{n},k}$, with $M_{n,k} = \{m,\ g(n, m, k) \neq 0\}$. According [13, Lemma 1.2] the set $M_{n,k} = \{m,\ g(n, m, k) \neq 0\}$ is finite for all n, k. ∎

For a function a and a discrete measure μ on K we denote the application of μ to a by $\mu(a) = \sum_k a(k)\mu(k)$ whenever the sum exists. Furthermore, for a function f and a measure μ we form the measure $f\mu$ by $f\mu(a) = \mu(fa)$ for all functions a on K.

Theorem 3.7 *If f, g are functions on K with finite support then $(f \star g)h = (fh) * (gh)$.*

PROOF: Let a be an arbitrary function on K. Application of Lemma 2.3 yields

$$\begin{aligned}
(f \star g)h(a) \;&=\; \sum_m a(m)(f \star g)(m)h(m) \;=\; \sum_m \sum_n a(m)f(n)(\overline{L_{\tilde{n}}}g)(m)h(m)h(n) \\
=\; \sum_n \sum_m f(n)(L_n a)(m)g(m)h(m)h(n) \;&=\; \sum_k \sum_{m,n} g(n, m, k)f(n)h(n)g(m)h(m)a(k) \\
=\; \sum_k (fh) * (gh)(k)a(k) \;&=\; (fh) * (gh)(a).
\end{aligned}$$
∎

If $\mathcal{K}$ is commutative, then $*$ is commutative and by the last lemma we see that then also $\star$ is commutative.

For a positive discrete measure σ on K and $1 \leq p < \infty$ we introduce the Banach spaces

$$\begin{aligned}
l^p(\sigma) \;&=\; \left\{f : K \to \mathbb{C},\ \sum_n |f(n)|^p \sigma(n) \;<\; \infty\right\}, \quad \|f\|_{p,\sigma} \;=\; \left(\sum_n |f(n)|^p \sigma(n)\right)^{1/p}, \\
l^\infty \;&=\; \left\{f : K \to \mathbb{C},\ \sup_n |f(n)| < \infty\right\}, \quad \|f\|_\infty \;=\; \sup_n |f(n)|.
\end{aligned}$$

Lemma 3.8 *If $f \in l^\infty$ then $L_n f \in l^\infty$ for all n and $\|L_n f\|_\infty \leq \gamma(n)\|f\|_\infty$.*

PROOF: For all n, m it holds

$$|L_n f(m)| \;=\; \left|\sum_k g(n, m, k)f(k)\right| \;\leq\; \sum_k |g(n, m, k)||f(k)| \;\leq\; \gamma(n)\|f\|_\infty.$$
∎

We now see that the sums in (7) converge if $f \in l^1(h), g \in l^\infty$ or $f \in l^\infty, g \in l^1(h)$, respectively, and Lemma 2.3 extends to these spaces.

Theorem 3.9 *The convolution* $\star$ *in (11) extends to* $l^1(\gamma h)\times l^1(h)$ *and* $\|f\star g\|_{1,h} \le \|f\|_{1,\gamma h}\|g\|_{1,h}$.

<u>Proof:</u> First, assume f, g to have finite support and a such that $a(k)(f\star g)(k) = |(f\star g)(k)|$. Theorem 3.7 yields

$$\begin{aligned}\|(f\star g)\|_{1,h} &= \sum_k |(f\star g)(k)|h(k) \;=\; (f\star g)h(a) \;=\; |(fh)*(gh)(a)| \\ &\le \sum_{n,m}\gamma(n)|g(m)||f(n)|h(n)h(m) \;=\; \|f\|_{1,\gamma h}\|g\|_{1,h}.\end{aligned}$$

Hence, $\star$ is continuous on a dense subspace of $l^1(\gamma h)\times l^1(h)$. Therefore, it can be uniquely continued. ∎

By using (9) the convolution extends quite analogous to $l^1(h)\times l^1(\tilde\gamma h)$ with $\|f\star g\|_{1,h} \le \|f\|_{1,h}\|g\|_{1,\tilde\gamma h}$. If $\mathcal{K}$ is bounded, i.e., $\gamma(n) \le M$ for all n, then the last theorem gives $\|f\star g\|_{1,h} \le M\|f\|_{1,h}\|g\|_{1,h}$. For $f\in l^1(h)$ define $L_f g = f\star g$. Clearly, L_f is then a bounded operator on $l^1(h)$ and $\|L_f\| \le M\|f\|_{1,h}$. With the norm $\|f\|' = \|L_f\|$ it holds $\|f\star g\|' \le \|f\|'\|g\|'$. Hence, if $\mathcal{K}$ is bounded, then $(l^1(h), \|\cdot\|', \star)$ is a Banach algebra.

Lemma 3.10 *For all* $f\in l^1(h)$ *and all* $m,n\in K$ *it holds*

$$(L_n f)(m) \;=\; h(\tilde n)^{-1}\overline{(\epsilon_{\tilde n}\star\overline{f})(m)} \tag{12}$$

and $\|L_n f\|_{1,h} \le \gamma(\tilde n)\|f\|_{1,h}$.

<u>Proof:</u> Since $f\in l^1(h)$ the right hand side of (12) exists by Theorem 3.9 and

$$h(\tilde n)^{-1}\overline{(\epsilon_{\tilde n}\star\overline{f})(m)} \;=\; h(\tilde n)^{-1}\sum_k \epsilon_{\tilde n}(k)(L_{\tilde k}f)(m)h(k) \;=\; (L_n f)(m).$$

Using Theorem 3.9 we further deduce

$$\begin{aligned}\|L_n f\|_{1,h} &= h(\tilde n)^{-1}\left\|\overline{\epsilon_{\tilde n}\star\overline{f}}\right\|_{1,h} \;\le\; h(\tilde n)^{-1}\|\epsilon_{\tilde n}\|_{1,\gamma h}\|f\|_{1,h} \\ &= h(\tilde n)^{-1}\sum_k \epsilon_{\tilde n}(k)\gamma(k)h(k)\;\|f\|_{1,h} \;=\; \gamma(\tilde n)\|f\|_{1,h}.\end{aligned}$$ ∎

Theorem 3.11 *The convolution* $\star$ *in (11) extends to* $l^1(\tilde\gamma h)\times l^\infty$. *It holds*

$$\|f\star g\|_\infty \;\le\; \|f\|_{1,\tilde\gamma h}\|g\|_\infty. \tag{13}$$

<u>PROOF:</u> Assume f, g to have finite support. By using Lemma 3.8 we obtain

$$\begin{aligned} |(f \star g)(m)| &= \left| \sum_n f(n) \overline{(L_{\tilde{n}} \overline{g})(m)} h(n) \right| \\ &\leq \sum_n |f(n)| \gamma(\tilde{n}) h(n) \|g\|_\infty = \|f\|_{1,\tilde{\gamma}h} \|g\|_\infty . \end{aligned}$$

Hence, $\star$ is bounded on a dense subspace of $l^1(\tilde{\gamma}h) \times l^\infty$ and can be extended. ■

By using $(f \star g)(m) = \sum_n (L_{\tilde{n}} \tilde{f})(m) \tilde{g}(n) h(n)$ we prove quite analogously that the convolution extends to $l^\infty \times l^1(\gamma\tilde{h})$ with $\|f \star g\|_\infty \leq \|f\|_\infty \|g\|_{1,\gamma\tilde{h}}$.

Theorem 3.12 *For* $1 \leq p \leq \infty$, *the convolution* $\star$ *in (11) extends to* $(l^1(\gamma h) \cap l^1(\tilde{\gamma}h)) \times l^p(h)$. *With* $1/p + 1/q = 1$ *it holds*

$$\|f \star g\|_{p,h} \leq \|f\|_{1,\gamma h}^{1/p} \|f\|_{1,\tilde{\gamma}h}^{1/q} \|g\|_{p,h}. \tag{14}$$

If $\mathcal{K}$ *is hermitian or commutative, then the inequality simplifies to* $\|f \star g\|_{p,h} \leq \|f\|_{1,\gamma h} \|g\|_{p,h}$.

<u>PROOF:</u> For $f \in l^1(\gamma h) \cap l^1(\tilde{\gamma}h)$ put $L_f g = f \star g$. By Theorem 3.9 it holds $\|L_f\|_{B(l^1(h))} \leq \|f\|_{1,\gamma h}$ where $B(l^1(h))$ denotes the Banach space of bounded operators from $l^1(h)$ into $l^1(h)$. Furthermore, by theorem 3.11 we have $\|L_f\|_{B(l^\infty)} \leq \|f\|_{1,\tilde{\gamma}h}$. Hence, inequality (14) is a consequence of the Riesz-Thorin interpolation theorem, see for example [21, p. 72]. If $\mathcal{K}$ is commutative then $h = \tilde{h}$ and $\gamma = \tilde{\gamma}$. ■

By defining an operator $R_g f = f \star g$ we derive quite analogously that the convolution $\star$ extends to $l^p(h) \times (l^1(\tilde{\gamma}h) \cap l^1(\gamma\tilde{h}))$ with $\|f \star g\|_{p,h} \leq \|f\|_{p,h} \|g\|_{1,\tilde{\gamma}h}^{1/p} \|g\|_{1,\gamma\tilde{h}}^{1/q}$.

Lemma 3.13 *For* $1 \leq p \leq \infty$ *and* $(1/p + 1/q = 1)$ *it holds* $\|L_n f\|_{p,h} \leq \gamma(\tilde{n})^{1/p} \gamma(n)^{1/q} \|f\|_{p,h}$.

<u>PROOF:</u> The proof is done by using (12) and Theorem 3.12. ■

Theorem 3.14 *Let* $1/p + 1/q = 1$. *For* $f \in l^p(h), g \in l^q(h)$ *it holds*

$$|(f \star g)(m)| \leq \gamma(\tilde{m})^{1/p} \gamma(m)^{1/q} \|f\|_{p,h} \|\tilde{g}\|_{q,h}. \tag{15}$$

<u>PROOF:</u> Applying Hölder's inequality in the second equation yields

$$\begin{aligned} |(f \star g)(m)| &= \left| \sum_n f(n) L_{\tilde{m}} \tilde{g}(n) h(n) \right| = \|f\|_{p,h} \|L_{\tilde{m}} \tilde{g}\|_{q,h} \\ &\leq \gamma(\tilde{m})^{1/p} \gamma(m)^{1/q} \|f\|_{p,h} \|\tilde{g}\|_{q,h} . \end{aligned}$$

■

If $\mathcal{K}$ is hermitian or commutative inequality (15) becomes $|(f \star g)(m)| \leq \gamma(m) \|f\|_{p,h} \|g\|_{q,h}$. In this case we introduce the Banach space

$$l^\infty(\gamma) = \left\{ f : K \to \mathbb{C}, \sup_n \frac{|f(n)|}{\gamma(n)} < \infty \right\}, \ \|f\|_{\infty,\gamma} = \sup_n \frac{|f(n)|}{\gamma(n)}. \tag{16}$$

Now, (15) becomes $\|f * g\|_{\infty,\gamma} \leq \|f\|_{p,h} \|g\|_{q,h}$.

4 Dual objects

We say that a generalized hypergroup $(\mathcal{K}', \mathcal{A}'_0)$ is a function realization, if $\mathcal{A}'_0$ is a dense subalgebra of the space $C(\mathcal{S})$, where $\mathcal{S}$ is a compact Hausdorff space. By using Gelfand theory, Obata and Wildberger proved that for commutative generalized hypergroups $(\mathcal{K}, \mathcal{A}_0)$ satisfying (B) there is an isomorphism $a \to a'$ onto a function realization $(\mathcal{K}', \mathcal{A}'_0)$. Moreover, there is a positive Radon measure μ on $\mathcal{S}$ with $\operatorname{supp}\mu = S$, $\mu(\mathcal{S}) = 1$ and

$$\phi_0(a) \;=\; \int_{\mathcal{S}} a'(x)d\mu(x) \qquad \text{for all } a \in \mathcal{A}_0,$$

and $\mathcal{K}'$ is a complete orthogonal set for $L^2(\mathcal{S}, \mu)$, see [13, Theorem 5.1].

From now on, we assume $(\mathcal{K}, \mathcal{A}_0)$ to be commutative and $\mathcal{A}_0$ to be a dense subalgebra of $C(\mathcal{S})$ for some compact Hausdorff space $\mathcal{S}$. The condition (B) now reads

$$\left|\int_{\mathcal{S}} c_n(x)|b(x)|^2 d\mu(x)\right| \;\le\; \kappa(n)\int_{\mathcal{S}} |b(x)|^2 d\mu(x) \;=\; \kappa(n)\|b\|^2_{L^2(\mathcal{S},\mu)} \quad \text{for all } b \in C(\mathcal{S}),$$

and therefore with

$$\kappa(n) \;=\; \|c_n\|_\infty \;=\; \sup_{x\in\mathcal{S}} |c_n(x)| \;<\; \infty$$

condition (B) is satisfied. The next lemma states that $\kappa(n)$ cannot be chosen smaller.

Lemma 4.1 *Let $(\mathcal{K}, \mathcal{A}_0)$ satisfy condition (B) with constants $\kappa(n)$. Then*

$$\sup_{x\in\mathcal{S}} |c_n(x)| \;\le\; \kappa(n) \qquad \textit{for all } n \in K.$$

In particular, it holds $\sup_{x\in\mathcal{S}} |c_n(x)| \le \gamma(n)$.

PROOF: Let us first remark that $L^2(\mathcal{S}, \mu)$ is the completion of $\mathcal{A}_0$ with respect to $\|\cdot\|_{2,\mu}$ since $\mathcal{K}$ is a complete orthogonal set for $L^2(\mathcal{S}, \mu)$. The inequality

$$\left|\int_{\mathcal{S}} c_n(x)|b(x)|^2 d\mu(x)\right| \;\le\; \kappa(n)\int_{\mathcal{S}} |b(x)|^2 d\mu(x) \tag{17}$$

is hence valid even for all $b \in L^2(\mathcal{S}, \mu)$. Now, let $x_0 \in \mathcal{S}$ and choose a family of neighborhoods $(V_i)_{i\in I}$ of x_0 such that $V_i \to \{x_0\}$. Further let $b_i = \chi_{V_i}/\|\chi_{V_i}\|_{2,\mu}$ where χ_{V_i} denotes the characteristic function of the set V_i. Clearly $\lim_i \int_{\mathcal{S}} c_n(x)|b_i(x)|^2 d\mu(x) = c_n(x_0)$. Since, $x_0 \in \mathcal{S}$ is arbitrarily chosen, inserting into (17) gives the assertion. Further, notice that $\kappa(n) = \gamma(n)$ is a valid choice by [13, Theorem 4.1]. ■

Now, let us consider dual objects of commutative generalized hypergroups. Obata and Wildberger already have defined characters [13, p. 74], but their definition seems to be too weak in order to develop harmonic analysis.

Definition 4.2 *We define two dual spaces by*

$$\begin{aligned}\mathcal{X}^b(K) &= \{\alpha \in l^\infty(\gamma),\ \alpha \neq 0,\ L_n\alpha(m) = \alpha(n)\alpha(m)\}\,,\\ \hat{K} &= \left\{\alpha \in \mathcal{X}^b(K),\ \alpha(\tilde{n}) \ = \ \overline{\alpha(n)}\right\}.\end{aligned}$$

The elements of $\mathcal{X}^b(K)$ *are called characters and the elements of* $\hat{K}$ *hermitian characters.*

Consider now an element x of $\mathcal{S}$. It is easily seen that $\alpha_x(n) = c_n(x)$ defines an element of $\hat{K}$. Hence $\hat{K} \neq \emptyset$. Since $\mathcal{A}_0$ is dense in $C(\mathcal{S})$ and $\mathcal{S}$ is a compact Hausdorff space it follows that for different $x, y \in \mathcal{S}$ we obtain different characters $\alpha_x \neq \alpha_y$, see also [13, Theorem 6.4]. Thus, we can identify $\mathcal{S}$ with a subset of $\hat{K}$ and we get the following inclusion relations

$$\mathcal{S} \ \subset \ \hat{K} \ \subset \ \mathcal{X}^b(K)\,. \tag{18}$$

The latter relation is well known for hypergroups and signed hypergroups. In contrast to the group case, these inclusions may be proper, as is illustrated by some known examples for hypergroups.
From $\alpha(n) = L_0\alpha(n) = \alpha(0)\alpha(n)$ it follows $\alpha(0) = 1$. Furthermore, since $\gamma(0) = 1$ it holds $\|\alpha\|_{\infty,\gamma} \geq 1$. By Lemma 4.1 $|c_n(x)| \ \leq \ \gamma(n)$, which implies $\|\alpha_x\|_{\infty,\gamma} = 1$ for all $x \in \mathcal{S}$.
For $r \geq 1$ let us define the following subsets of the duals

$$\begin{aligned}\mathcal{X}^b_r(K) &= \left\{\alpha \in \mathcal{X}^b(K),\ \|\alpha\|_{\infty,\gamma} \leq r\right\},\\ \hat{K}_r &= \left\{\alpha \in \hat{K},\ \|\alpha\|_{\infty,\gamma} \leq r\right\}.\end{aligned}$$

If $\mathcal{K}$ is bounded then $\mathcal{X}^b(K) \ = \ \mathcal{X}^b_R(K)$ and $\hat{K} = \hat{K}_R$, where $R = \sup_n \gamma(n) \leq \infty$. In fact, in that case $l^\infty(\gamma) = l^\infty$ setwise and for a character $\alpha \in l^\infty$ it holds

$$|\alpha(n)|^2 \ = \ |\alpha(n)\alpha(n)| \ = \ |L_n\alpha(n)| \ \leq \ \gamma(n)\|\alpha\|_\infty\,. \tag{19}$$

Taking the supremum over all $n \in K$ yields $\|\alpha\|_\infty \leq \sup_n \gamma(n)$. Since $\gamma(n) \geq 1$ we further deduce

$$1 \ \leq \ \|\alpha\|_{\infty,\gamma} \ \ \leq \ \ \|\alpha\|_\infty \ \leq \ R. \tag{20}$$

We equip $\mathcal{X}^b(K)$ with the topology of pointwise convergence and subsets of $\mathcal{X}^b(K)$ with the induced topologies. With these topologies the functions $s_n : \mathcal{X}^b(K) \to \mathbb{C}$, $s_n(\alpha) = \alpha(n)$ and their restrictions to the other duals are continuous. We only state without a proof that the Gelfand topology on $\mathcal{S}$ is the topology induced by $\mathcal{X}^b(K)$, i.e., the topology of pointwise convergence.

5 Fourier transform

Now, due to our dual objects we are able to perform some Fourier analysis in the context of commutative SBG hypergroups.

Definition 5.1 *For $\mu \in M_\gamma(K)$ we introduce the following two versions of the Fourier-Stieltjes-transform by*

$$\begin{aligned}\hat{\mu}(\alpha) &= \sum_n \alpha(n)\mu(n) \qquad \text{for } \alpha \in \hat{K},\\ \mathcal{F}(\mu)(\alpha) &= \sum_n \alpha(n)\mu(n) \qquad \text{for } \alpha \in \mathcal{X}^b(K).\end{aligned}$$

For $x \in \mathcal{S} \subset \hat{K}$ we write $\hat{\mu}(x) = \hat{\mu}(\alpha_x) = \sum_n c_n(x)\mu(n)$.

The following lemma states that our definition makes sense.

Lemma 5.2 *If $\alpha \in \mathcal{X}_r^b(K)$ then $|\mathcal{F}(\mu)(\alpha)| \le r\|\mu\|_\gamma$ and $\mathcal{F}(\mu)$ is a continuous function from $\mathcal{X}_r^b(K)$ into $\mathbb{C}$.*

<ins>Proof:</ins> Let $\alpha \in \mathcal{X}_r^b(K)$, i.e., $|\alpha(n)| \le r\gamma(n)$ for all n. We obtain

$$|\mathcal{F}(\mu)(\alpha)| \le \sum_n |\alpha(n)||\mu(n)| \le r\sum_n |\gamma(n)||\mu(n)| = r\|\mu\|_\gamma .$$

Since the functions $s_n(\alpha) = \alpha(n)$ are continuous on $\mathcal{X}_r^b(K)$ for fixed n it follows that $\mathcal{F}(\mu)$ is continuous on $\mathcal{X}_r^b(K)$. ■

Definition 5.3 *For $f \in l^1(\gamma h)$ we define two versions of the Fourier transform by*

$$\begin{aligned}\hat{f}(\alpha) &= \widehat{(fh)}(\alpha) = \sum_n f(n)\alpha(n)h(n) \qquad \text{for } \alpha \in \hat{K},\\ \mathcal{F}(f)(\alpha) &= \mathcal{F}(fh)(\alpha) = \sum_n f(n)\alpha(n)h(n) \qquad \text{for } \alpha \in \mathcal{X}^b(K).\end{aligned}$$

For $x \in \mathcal{S}$ we write $\hat{f}(x) = \hat{f}(\alpha_x) = \sum_n f(n)c_n(x)h(n)$.

By interpreting measures on K as functions on K we clearly have $l^1(\gamma h) = \{f, fh \in M_\gamma(K)\}$ and hence, Lemma 5.2 immediately implies that the Fourier transform is continuous on $\mathcal{X}_r^b(K)$ for all $r \ge 1$ and for $\alpha \in \mathcal{X}_r^b(K)$ it holds

$$|\hat{f}(\alpha)| \le \|f\|_{1,\gamma h}. \tag{21}$$

In order to define the Fourier transform for $f \in l^2(h)$ we remark that $\{\sqrt{h(n)}c_n, n \in K\}$ is a complete orthonormal set for $L^2(\mathcal{S},\mu)$, see [13, Corollary 3.4]. Therefore, the series $\sum_n f(n)c_nh(n)$ converges in $L^2(\mathcal{S},\mu)$ by Parseval's identity

$$\int_{\mathcal{S}} |\sum_n f(n)c_n(x)h(n)|^2 d\mu(x) = \left\|\sum_n f(n)c_nh(n)\right\|_{2,\mu}^2 = \sum_n |f(n)|^2h(n) = \|f\|_{2,h}^2. \tag{22}$$

Hence, we define the Fourier transform of $f \in l^2(h)$ by

$$\hat{f} = \sum_n f(n)c_nh(n)$$

where convergence of the sum is understood in $L^2(\mathcal{S},\mu)$. In (22) we already proved Plancherel's theorem.

Theorem 5.4 *The Fourier transform is an isometric isomorphism from $l^2(h)$ into $L^2(\mathcal{S},\mu)$, in particular for $f \in l^2(h)$ it holds $\|\hat{f}\|_{2,\mu} = \|f\|_{2,h}$.*

As a consequence of Plancherel's theorem we obtain a uniqueness theorem for the Fourier transform on $l^1(\gamma h)$.

Theorem 5.5 *If $f \in l^1(\gamma h)$ and $\mathcal{F}(f)_{|\mathcal{S}} = 0$ then $f = 0$.*

PROOF: Let $f \in l^1(\gamma h)$. Since $\gamma(n) \geq 1$ we have $f \in l^1(h)$. Now denote $N = \{n \in K,\ |f(n)| \geq 1\}$. Since $\gamma(n) \geq h(n)^{-1}$ this set is finite. We obtain

$$\sum_{n\in K} |f(n)|^2 h(n) \quad \leq \quad \sum_{n\in N} |f(n)|^2 h(n) + \|f\|_{1,h} \ < \ \infty,$$

which means $f \in l^2(h)$. The Fourier transform on $l^1(\gamma h)$ coincides with the one on $l^2(h)$ μ-almost everywhere and by Plancherel's theorem $\|f\|_{2,h} = \|\hat{f}\|_{2,\mu} = \|\mathcal{F}(f)_{|\mathcal{S}}\|_{2,\mu} = 0$. We therefore obtain $f = 0$. ■

Let us turn our attention now to the relation of Fourier transform and convolution.

Theorem 5.6 *If $f, g \in l^1(\gamma h)$ such that $f \star g \in l^1(\gamma h)$ then*

$$\mathcal{F}(f \star g)(\alpha) \quad = \quad \mathcal{F}(f)(\alpha)\mathcal{F}(g)(\alpha) \qquad \textit{for all } \alpha \in \mathcal{X}^b(K)\,. \tag{23}$$

PROOF: We use Lemma 2.3 and Fubini's theorem to obtain

$$\begin{aligned} \mathcal{F}(f \star g)(\alpha) &= \sum_n (f \star g)(n)\alpha(n)h(n) \;=\; \sum_n \sum_m f(m)(\overline{L_{\tilde{m}}}g)(n)h(m)\alpha(n)h(n) \\ &= \sum_m \sum_n (L_m\alpha)(n)g(n)h(n)f(m)h(m) \;=\; \sum_m \sum_n g(n)\alpha(n)h(n)f(m)\alpha(m)h(m) \\ &= \sum_n g(n)\alpha(n)h(n) \sum_m f(m)\alpha(m)h(m) \;=\; \mathcal{F}(f)(\alpha)\mathcal{F}(g)(\alpha). \end{aligned}$$
■

Corollary 5.7 *The convolution $\star$ extends to $l^1(\gamma h) \times l^1(\gamma h) \to l^2(h)$. It holds*

$$\|f \star g\|_{2,h} \ \leq \ \|f\|_{1,\gamma h}\|g\|_{1,\gamma h}. \tag{24}$$

PROOF: First suppose $f, g \in l^1(\gamma h)$ such that $f \star g \in l^1(\gamma h)$. Using Plancherel's theorem 5.4, Theorem 5.6 and (21) we obtain

$$\|f \star g\|_{2,h} \quad = \quad \|\widehat{f \star g}\|_{2,\mu} \ = \ \|\hat{f}\hat{g}\|_{2,\mu} \ \leq \ \|\hat{f}\|_{\infty,\mathcal{S}}\|\hat{g}\|_{\infty,\mathcal{S}} \ \leq \ \|f\|_{1,\gamma h}\|g\|_{1,\gamma h}.$$

Hence, the convolution is continuous on $C = \{(f,g),\ f,g \in l^1(\gamma h), f \star g \in l^1(\gamma h)\}$. Since functions of finite support are dense in $l^1(\gamma h)$ and the convolution of two such functions has again finite support, we see that C is dense in $l^1(\gamma h) \times l^1(\gamma h)$. Thus $\star$ uniquely extends to $l^1(\gamma h) \times l^1(\gamma h)$ and (24) holds. ■

Note, that implicitly we used the commutativity of $\mathcal{K}$ in this proof. Immediately, we obtain that the convolution Theorem 5.6 holds for all $f, g \in l^1(\gamma h)$ with the slight adjustment that in general (23) holds only for μ-almost all $\alpha \in \mathcal{X}^b(K)$.

An involution on $l^1(h)$ is given by $f^*(n) = \overline{f(\tilde{n})}$, which is preserved by the Fourier transform on $\hat{K}$, i.e.,

$$\widehat{f^*}(\alpha) = \sum_n \overline{f(\tilde{n})}\alpha(n)h(n) = \sum_n \overline{f(\tilde{n})\alpha(\tilde{n})}h(n) = \overline{\hat{f}(n)} \quad \text{for all } \alpha \in \hat{K}.$$

The inverse Fourier transform for $F \in L^1(\mathcal{S}, \mu)$ is defined by

$$\check{F}(n) = \int_{\mathcal{S}} F(x)\overline{c_n(x)}d\mu(x), \quad \text{for all } n \in K.$$

We can even extend this definition to a larger space. Let $M(\mathcal{S})$ denote the space of complex bounded Radon measures on $\mathcal{S}$ with the total variation as norm. For $\rho \in M(\mathcal{S})$ we define the inverse Fourier-Stieltjes transform by

$$\check{\rho}(n) = \int_{\mathcal{S}} \overline{c_n(x)}d\rho(x), \quad \text{for all } n \in K.$$

Clearly, $(F\mu)\check{} = \check{F}$. Parseval's identity immediately gives the following inversion theorem.

Theorem 5.8 *(i) If $f \in l^2(h)$ then $(\hat{f})\check{} = f$.*

(ii) If $F \in L^2(\mathcal{S}, \mu)$ then $\widehat{(\check{F})} = F$ μ-almost everywhere.

Theorem 5.9 *For the inverse Fourier-Stieltjes transform the following is true.*

(i) For $\rho \in M(\mathcal{S})$ we have $\check{\rho} \in l^\infty(\gamma)$ and $\|\check{\rho}\|_{\infty,\gamma} \le \|\rho\|$.

(ii) For $F \in L^1(\mathcal{S}, \mu)$ it holds $\|\check{F}\|_{\infty,\gamma} \le \|F\|_{1,\mu}$.

(iii) For $F \in L^1(\mathcal{S}, \mu)$ we have $\check{F} \in c_0(\gamma)$ where $c_0(\gamma)$ denotes the closure with respect to $\|\cdot\|_{\infty,\gamma}$ of the set of all functions with finite support. Furthermore, the image of the inverse Fourier transform of $L^1(\mathcal{S}, \mu)$ is dense in $c_0(\gamma)$.

<u>Proof:</u> (i) For $\rho \in M(\mathcal{S})$ and $n \in K$ we have

$$|\check{\rho}(n)| = \left|\int_{\mathcal{S}} \overline{c_n}(x)d\rho(x)\right| \le \|c_n\|_{\infty,\mathcal{S}}\|\rho\| \le \gamma(n)\|\rho\|.$$

The statement (ii) is an easy consequence of (i) by observing $(F\mu)\check{} = \check{F}$ and $\|F\|_{1,\mu} = \|F\mu\|$.
(iii) Let $\epsilon > 0$ and choose $G \in C(\mathcal{S})$ such that $\|F - G\|_{1,\mu} \le \epsilon/2$. Since $C(\mathcal{S}) \subset L^2(\mathcal{S}, \mu)$ it holds $\check{G} \in l^2(h)$. Hence, there exist ϕ with $|\operatorname{supp}\phi| < \infty$ such that $\|\check{G} - \phi\|_{2,h} \le \epsilon/2$. Using $\gamma(n) \ge \max\{1, h(n)^{-1}\}$ we deduce for arbitrary $f \in l^2(h)$ that

$$\frac{|f(n)|^2}{\gamma(n)^2} \le \frac{|f(n)|^2}{\gamma(n)} \le |f(n)|^2 h(n) \le \sum_n |f(n)|^2 h(n) = \|f\|_{2,h}^2,$$

yielding $\|f\|_{\infty,\gamma} \leq \|f\|_{2,h}$. Note that we hereby derived $l^2(h) \subset l^\infty(\gamma)$. Now, using this estimation we obtain $\|\check{G}-\phi\|_{\infty,\gamma} \leq \|\check{G}-\phi\|_{2,h} \leq \epsilon/2$ and further

$$\begin{aligned} |\check{F}(k)-\phi(k)| &\leq |\check{F}(k)-\check{G}(k)| + |\check{G}(k)-\phi(k)| \\ &\leq \gamma(k)(\|F-G\|_{1,\mu}+\epsilon/2) \leq \epsilon\gamma(k)\,, \end{aligned}$$

which is equivalent to $\|\check{F}-\phi\|_{\infty,\gamma} \leq \epsilon$. Hence, $\check{F}$ can be approximated with respect to $\|\cdot\|_{\infty,\gamma}$ by functions with finite support. Since all function with finite support are contained in the image of the inverse Fourier transform of $L^1(\mathcal{S},\mu)$ the image of $L^1(\mathcal{S},\mu)$ is dense in $c_0(\gamma)$. ■

Observe, that the last result generalizes the Riemann-Lebesgue lemma. We also have a uniqueness theorem for the inverse Fourier transform.

Theorem 5.10 *Let $\rho \in M(\mathcal{S})$. If $\check{\rho} = 0$ then $\rho = 0$.*

PROOF: Assume that $\rho \neq 0$ but $\check{\rho} = 0$. By [13, Theorem 5.1] $\mathcal{A}_0 = \{\hat{f}_{|\mathcal{S}},\ |\operatorname{supp} f| < \infty\}$ is a dense subalgebra of $C(\mathcal{S})$. Hence, there is some f with finite support such that

$$\int_{\mathcal{S}} \hat{f}(x)d\rho(x) \neq 0.$$

However, we have

$$\int_{\mathcal{S}} \hat{f}(x)d\rho(x) = \sum_n f(n)\int_{\mathcal{S}} \overline{c_{\tilde{n}}(x)}d\rho(x)h(n) = \sum_n f(n)\check{\rho}(\tilde{n})h(n) = 0.$$ ■

Denoting $p_n(x) = \epsilon_n(x)/h(n)$ we have $\hat{p}_n(x) = c_n(x)$ and $(\hat{p}_n)^\vee = p_n$ yielding

$$\check{\mu}(n) = \int_{\mathcal{S}} \overline{c_n(x)}d\mu(x) = p_{\tilde{n}}(0) = \epsilon_0(\tilde{n}),$$

i.e., $\check{\mu} = \epsilon_0$. Another important property was shown in the proof of Theorem 5.10 above. Suppose f has finite support and $\rho \in M(\mathcal{S})$. Then

$$\int_{\mathcal{S}} \hat{f}(x)d\rho(x) = \sum_n f(n)\check{\rho}(\tilde{n})h(n). \tag{25}$$

We can extend the uniqueness theorem to the following result.

Theorem 5.11 *Let $f \in l^1(\gamma h)$ and $\rho \in M(\mathcal{S})$. It holds $\check{\rho} = f$ if and only if $\rho = \hat{f}\mu$.*

PROOF: For $\rho = \hat{f}\mu$ we already know by Theorem 5.8 that $\check{\rho} = (\hat{f})^\vee = f$. Now suppose $f = \check{\rho}$ and let g have finite support. With (25) and $(\hat{g})^\vee = g$ we obtain

$$\begin{aligned} \int_{\mathcal{S}} \hat{g}(x)\hat{f}(x)d\mu(x) &= \int_{\mathcal{S}} \hat{g}(x)\sum_n f(n)c_n(x)h(n)d\mu(x) = \sum_n f(n)\int_{\mathcal{S}} \hat{g}(x)c_n(x)d\mu(x)h(n) \\ &= \sum_n f(n)\int_{\mathcal{S}} \hat{g}(x)\overline{c_{\tilde{n}}(x)}d\mu(x)h(n) \\ &= \sum_n f(n)g(\tilde{n})h(n) = \sum_n g(n)\check{\rho}(\tilde{n})h(n) = \int_{\mathcal{S}} \hat{g}(x)d\rho(x). \end{aligned}$$

Since $\{\hat{g}_{|\mathcal{S}}, |\operatorname{supp} g| < \infty\}$ is dense in $C(\mathcal{S})$ we see that $\rho = \hat{f}\mu$. ■

A rewriting of the last result gives the inversion theorem.

Theorem 5.12 *The following two inversion formulas hold.*

(i) Let $f \in l^1(\gamma h)$. Then for every $n \in K$ it holds

$$f(n) \;=\; \int_{\mathcal{S}} \hat{f}(x)\overline{c_n(x)}d\mu(x)\,.$$

(ii) Let $F \in L^1(\mathcal{S},\mu)$ such that $\check{F} \in l^1(\gamma h)$. Then for μ-almost every $x \in \mathcal{S}$ it holds

$$F(x) \;=\; \sum_n \check{F}(n)c_n(x)h(n)\,. \tag{26}$$

If in addition F is continuous, then (26) holds for all $x \in \mathcal{S}$.

<u>Proof:</u> (i) follows by Theorem 5.8(i). For (ii) put $\rho = F\mu$. Then $\check{\rho} = \check{F} \in l^1(\gamma h)$. With Theorem 5.11 it holds $\rho = (\check{F})\hat{}\,\mu$ which is equivalent to $F = (\check{F})\hat{}$ in $L^1(\mathcal{S},\mu)$. Since the right hand side of (26) is continuous, equality holds for all $x \in \mathcal{S}$ if F is continuous. ■

6 Orthogonal polynomials on the real line

Let μ be a probability measure on the real line. We denote the support of μ by $\mathcal{S}$ and assume $\operatorname{card}(\mathcal{S}) = \infty$. Furthermore, let $(P_n)_{n=0}^{\infty}$ denote an orthogonal polynomial sequence with respect to μ, that is $\int P_n P_m \,d\mu \neq 0$ if and only if $n = m$. The polynomials P_n are assumed to have real coefficients with $\deg(P_n) = n$ and $P_0 = 1$. It is well known that the sequence $(P_n)_{n\in\mathbb{N}_0}$ satisfies a three term recurrence relation of the following type

$$P_1(x)P_n(x) = a_nP_{n+1}(x) + b_nP_n(x) + c_nP_{n-1}(x), \quad n \geq 1, \tag{27}$$

with $P_0(x) = 1$ and $P_1(x) = (x-b)/a$, where the coefficients are real numbers with $c_1 > 0$, $c_na_{n-1} > 0$, $n > 1$. Conversely, if we define $(P_n)_{n=0}^{\infty}$ by (27) there is a measure μ with the assumed properties, see [3].
The linearization coefficients $g(n,m,k)$ are uniquely defined by

$$P_nP_m \;=\; \sum_{k=0}^{\infty} g(n,m,k)P_k \;=\; \sum_{k=|n-m|}^{n+m} g(n,m,k)P_k. \tag{28}$$

The linearization coefficients are obtained recursively based on the coefficients of the tree term recurrence relation.

Lemma 6.1 *We have $g(0,m,m) = 1$. In case $m \geq 1$ we get $g(1,m,m-1) = c_m$, $g(1,m,m) = b_m$ and $g(1,m,m+1) = a_m$. In case $m \geq n \geq 2$ we get the recurrence relation:*

(i)

$$g(n,m,n+m) = g(n-1,m,n+m-1)\frac{a_{n+m-1}}{a_{n-1}} = \frac{a_m a_{m+1}\cdots a_{n+m-1}}{a_1 a_2 \cdots a_{n-1}}, \quad (29)$$

$$g(n,m,m-n) = g(n-1,m,m-n+1)\frac{c_{m-n+1}}{a_{n-1}} = \frac{c_m c_{m-1}\cdots c_{m-n+1}}{a_1 a_2 \cdots a_{n-1}}, \quad (30)$$

(ii)

$$\begin{aligned} g(n,m,n+m-1) &= g(n-1,m,n+m-1)\frac{b_{n+m-1}-b_{n-1}}{a_{n-1}} \\ &+ g(n-1,m,n+m-2)\frac{a_{n+m-2}}{a_{n-1}}, \\ g(n,m,m-n+1) &= g(n-1,m,m-n+1)\frac{b_{m-n+1}-b_{n-1}}{a_{n-1}} \\ &+ g(n-1,m,m-n+2)\frac{c_{m-n+2}}{a_{n-1}}, \end{aligned}$$

(iii) For $k = 2, 3, \dots, 2n-2$ it holds

$$\begin{aligned} g(n,m,m-n+k) &= g(n-1,m,m-n+k-1)\frac{a_{m-n+k-1}}{a_{n-1}} \\ &+ g(n-1,m,m-n+k)\frac{b_{m-n+k}-b_{n-1}}{a_{n-1}} \\ &+ g(n-1,m,m-n+k+1)\frac{c_{m-n+k+1}}{a_{n-1}} \\ &- g(n-2,m,m-n+k)\frac{c_{n-1}}{a_{n-1}}. \end{aligned}$$

<u>Proof:</u> In case $m \geq n \geq 2$ we have

$$P_n = \frac{1}{a_{n-1}} P_1 P_{n-1} - \frac{b_{n-1}}{a_{n-1}} P_{n-1} - \frac{c_{n-1}}{a_{n-1}} P_{n-2}.$$

So

$$\begin{aligned} P_n P_m &= \sum_{k=m-n+1}^{m+n-1} g(n-1,m,k)\left(\frac{a_k}{a_{n-1}}P_{k+1} + \frac{b_k}{a_{n-1}}P_k + \frac{c_k}{a_{n-1}}P_{k-1}\right) \\ &-\frac{b_{n-1}}{a_{n-1}}\sum_{m-n+1}^{m+n-1} g(n-1,m,k)P_k - \frac{c_{n-1}}{a_{n-1}}\sum_{m-n+2}^{m+n-2} g(n-2,m,k)P_k, \end{aligned}$$

which implies the recurrence formulas (i)-(iii). The second equations in (i) are proven by induction. ■

We easily derive

$$h(n) = g(n,n,0)^{-1} = \left(\int P_n^2(x)\,d\mu(x)\right)^{-1} = \frac{\prod_{i=1}^{n-1} a_i}{\prod_{i=1}^{n} c_i}. \quad (31)$$

Let $\mathcal{K} = \{P_n, n \in \mathbb{N}_0\}$, $\mathcal{A}_0$ be the set of polynomials with complex coefficients in one real variable and * be the complex conjugation $^-$.

Theorem 6.2 *We have the following classification.*

(i) $(\mathcal{K}, \mathcal{A}_0)$ *is a hermitian and commutative generalized hypergroup.*

(ii) $(\mathcal{K}, \mathcal{A}_0)$ *satisfies property (B) if and only if* $\mathcal{S}$ *is compact.*
$\mathcal{S}$ *is compact if and only if the sequences* $(c_n a_{n-1})$ *and* (b_n) *are bounded.*

(iii) $(\mathcal{K}, \mathcal{A}_0)$ *is an SBG hypergroup if and only if the sequences* (a_n)*,* (b_n) *and* (c_n) *are bounded.*

<u>Proof:</u> For (i) and (ii) see [3] and [13].
If $\mathcal{K}$ is semi-bounded then there is a bound for $g(1,n,n+1) = a_n$, $g(1,n,n) = b_n$ and $g(1,n,n-1) = c_n$. Let $|a_n|, |b_n|, |c_n| < B$. It is sufficient to prove $|g(n,m,k)| < M_n$ for all $m, k \in \mathbb{N}_0$, which implies $\sum_{k=|n-m|}^{n+m} |g(n,m,k)| \le (2n+1)M_n$ for all $m \in \mathbb{N}_0$. $M_0 = 1$ and $M_1 = B$ is a proper choice. Now let us assume that for $n \ge 2$ exist proper $M_0, M_1, \ldots, M_{n-1}$. According to the recurrence relation of the linearization coefficients, see Lemma 6.1, we get

$$|g(n,m,k)| \;\le\; \frac{4B}{|a_{n-1}|} M_{n-1} + \frac{B}{|a_{n-1}|} M_{n-2} \;=\; M_n. \qquad \blacksquare$$

Therefore we call $\mathcal{K}$ a generalized polynomial hypergroup or an SBG polynomial hypergroup, respectively. In order to get normalized generalized hypergroups Obata and Wildberger have investigated renormalizations in [13]. The following lemma shows that there always exist a renormalization of a generalized polynomial hpergroup $\mathcal{K} = \{P_n; n \in \mathbb{N}_0\}$ with property (B) which is semi-bounded.

Lemma 6.3 *Suppose* μ *to have compact support* $\mathcal{S}$*. Then the monic polynomials* Q_n *and the orthonormal polynomials* $p_n = \sqrt{h(n)} P_n$ *with respect to* μ *constitute an SBG polynomial hypergroup.*

<u>Proof:</u> Let the monic polynomials be defined by $Q_0 = 1$, $Q_1(x) = x - b'$ and $Q_1 Q_n = Q_{n+1} + b'_n Q_n + c'_n Q_{n-1}$, $n \ge 1$, where $c'_n > 0$. Since μ has compact support, (b'_n) and (c'_n) are bounded sequences. By Theorem 6.2 (iii) the corresponding generalized hypergroup is semi-bounded. Now, it is simple to derive that the corresponding orthonormal polynomials are defined by $p_0 = 1$, $p_1 = (x - b')/\sqrt{c'_1}$ and $p_1 p_n = a''_n p_{n+1} + b'_n/\sqrt{c'_1} p_n + a''_{n-1} p_{n-1}$, where $a''_n = \sqrt{c'_{n+1}/c'_1}$. Since (c'_n) is bounded again by Theorem 6.2 (iii) the corresponding generalized hypergroup is semi-bounded. $\blacksquare$

Now, we are looking for an OPS $(R_n)_{n\in\mathbb{N}_0}$ with $\sum_k g^R(n,m,k) = 1$ for all $n, m \in \mathbb{N}_0$, which is equivalent to the existence of $x_0 \in \mathbb{R}$ with $R_n(x_0) = 1$ for all $n \in \mathbb{N}_0$.

Theorem 6.4 *Suppose* μ *to have compact support and let* $(P_n)_{n\in\mathbb{N}_0}$ *be an arbitrary orthogonal polynomial sequence with respect to* μ*. Denote by* $[d,e]$ *the smallest interval containing* $\mathcal{S}$*. Choose* $x_0 \in \mathbb{R} \setminus (d,e)$ *and define* $R_n(x) = P_n(x)/P_n(x_0)$*,* $n \in \mathbb{N}_0$*. Then* $\mathcal{K} = \{R_k, k \in \mathbb{N}_0\}$ *is a normalized SBG hypergroup.*

<u>PROOF:</u> Let $(Q_n)_{n\in\mathbb{N}_0}$ be the monic orthogonal polynomials with respect to μ as in the proof of Lemma 6.3. Then

$$R_1 R_n \;=\; \frac{Q_{n+1}(x_0)}{Q_1(x_0)Q_n(x_0)}R_{n+1} + \frac{b'_n}{Q_1(x_0)}R_n + \frac{c'_n Q_{n-1}(x_0)}{Q_1(x_0)Q_n(x_0)}R_{n-1}, \quad n \geq 1.$$

Let $a'''_n = \frac{Q_{n+1}(x_0)}{Q_1(x_0)Q_n(x_0)}$, $b'''_n = \frac{b'_n}{Q_1(x_0)}$ and $c'''_n = \frac{c'_n Q_{n-1}(x_0)}{Q_1(x_0)Q_n(x_0)}$. Since (b'_n) is bounded $(b'''_n)_{n\in\mathbb{N}}$ is bounded, too. By [3, p. 110, Theorem 2.4] for $x_0 \notin (d,e)$ we have

$$0 \;<\; \frac{Q_{n+1}(x_0)}{(x_0 - b'_n - b')Q_n(x_0)} \;\leq\; 1, \quad n \geq 0.$$

Hence, $|a'''_n| < |(x_0 - b'_n - b')/Q_1(x_0)|$, which shows the boundedness of (a'''_n). Finally, $a'''_n + b'''_n + c'''_n = 1$ yields the boundedness of (c'''_n). By Theorem 6.2 (iii) the proof is complete. ■

Now, let us examine the duals of an SBG polynomial hypergroup. We define the sets

$$D_r \;=\; \{z \in \mathbb{C},\ |P_n(z)| \leq r\gamma(n) \text{ for all } n \in \mathbb{N}_0\}, \quad D \;=\; \bigcup_{r\geq 1} D_r, \tag{32}$$

$$D_r^s \;=\; D_r \cap \mathbb{R} \quad \text{and} \quad D^s \;=\; \bigcup_{r\geq 1} D_r^s. \tag{33}$$

Furthermore we define for some $z \in \mathbb{C}$ the function $\alpha_z(n) = P_n(z)$ for all $n \in \mathbb{N}_0$. Then the following theorem holds.

Theorem 6.5 *Let $\mathcal{K} = \{P_n, n \in \mathbb{N}_0\}$ be an SBG polynomial hypergroup.*

(i) It holds $\mathcal{X}^b(\mathbb{N}_0) = \{\alpha_z, z \in D\}$ and $\widehat{\mathbb{N}_0} = \{\alpha_x, x \in D^s\}$.

(ii) The mappings

$$D \to \mathcal{X}^b(\mathbb{N}_0), \quad z \mapsto \alpha_z \quad \text{and} \quad D^s \to \widehat{\mathbb{N}_0}, \quad x \mapsto \alpha_x$$

are homeomorphisms.

(iii) $\mathcal{X}^b(\mathbb{N}_0)$ and $\widehat{\mathbb{N}_0}$ are bounded.

<u>PROOF:</u> (i) For $z \in D_r$ it holds $\|\alpha_z\|_{\infty,\gamma} \leq r$, $\alpha_z \neq 0$, see [3, I Theorem 5.3], and $L_n\alpha_z(m) = \alpha_z(n)\alpha_z(m)$, hence $\{\alpha_z, z \in D_r\} \subset \mathcal{X}_r^b(\mathbb{N}_0)$.
Now suppose $\alpha \in \mathcal{X}_r^b(\mathbb{N}_0)$ and put $z = a_0\alpha(1) + b_0$. We obtain $\alpha(1)\alpha(n) = L_1\alpha(n) = a_n\alpha(n+1) + b_n\alpha(n) + c_n\alpha(n-1)$. Since $\alpha(0) = 1$ and $\alpha(1) = (z - b_0)/a_0$, $\alpha(n)$ satisfies the same recurrence relation as $P_n(z)$, hence they must be equal. This yields $\mathcal{X}_r^b(\mathbb{N}_0) \subset \{\alpha_z, z \in D_r\}$
Note that $P_n(z) = \overline{P_n(z)}$ for all $n \in \mathbb{N}_0$ implies $z \in \mathbb{R}$.

(ii) Let $V(\alpha_{z_0}, \epsilon, n_1, \ldots, n_k) = \left\{\alpha \in \mathcal{X}^b(K), |\alpha(n_i) - \alpha_{z_0}(n_i)| < \epsilon, i = 1, \ldots, k\right\}$. Clearly, its inverse under the mapping $z \mapsto \alpha_z$ is the set $\bigcap_{i=1}^k \{z \in D,\ |P_{n_i}(z) - P_{n_i}(z_0)| < \epsilon\}$, which is open. Since $\mathcal{X}^b(\mathbb{N}_0)$ is equipped with the topology of pointwise convergence the mapping

$\mathcal{X}^b(\mathbb{N}_0) \to D$, $\alpha_z \mapsto a_0\alpha_z(1) + b_0$ is continuous, too . The second statement follows since $\widehat{\mathbb{N}_0}$ bears the induced topology.

(iii) Denote by B the bound of $(|a_n|)$, $(|b_n|)$ and $(|c_n|)$, and choose $M > 0$ such that the zeros $z_{n,1}, z_{n,2}, \dots, z_{n,n}$ of any P_n are elements of the interval $[-M, M]$. We have $P_n(z) = \alpha_n \prod_{i=1}^n (z - z_{n,i})$ with $\alpha_n = (a^n \prod_{i=1}^{n-1} a_i)^{-1}$. Choose $z \in D$ and assume $|z| > M$. Then there exists $r \geq 1$ such that $|\alpha_n| \prod_{i=1}^n |z - z_{n,i}| \leq r\gamma(n)$. Since $|z - z_{n,i}| \geq |z| - M$ we get $(|z| - M)^n \leq r\gamma(n)/\alpha_n$. By Lemma 6.1 we are able to deduce $\gamma(n) = \mathcal{O}(n(|a|B)^n|\alpha_n|)$. Therefore there exists $C > 0$ such that $|z| - M \leq |a|B \sqrt[n]{rCn}$ for all $n \in \mathbb{N}$, which implies $|z| \leq M + |a|B$. ■

We would like to mention the the question wether the dual of an SBG polynomial hypergroup is compact is still open.

7 Jacobi polynomials

The Jacobi polynomials $P_n^{(\alpha,\beta)}$ are orthogonal with respect to the measure

$$\pi(x) = (1-x)^\alpha(1+x)^\beta dx, \quad \text{for all } \alpha, \beta > -1,$$

with $\operatorname{supp} \pi = [-1, 1] = \mathcal{S}$. According to Theorem 6.4 they form a normalized SBG polynomial hypergroup when normalizing at a point $x_0 \notin (-1, 1)$.
In case $x_0 = 1$ the three term recurrence relation coefficients are given by

$$\begin{aligned} a &= \frac{2(\alpha+1)}{\alpha+\beta+2}, \qquad b = \frac{\beta-\alpha}{\alpha+\beta+2}, \\ a_n &= \frac{(n+\alpha+\beta+1)(n+\alpha+1)(\alpha+\beta+2)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+1)(\alpha+1)}, \\ b_n &= \frac{\alpha-\beta}{2(\alpha+1)}\left[1 - \frac{(\alpha+\beta+2)(\alpha+\beta)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta)}\right], \\ c_n &= \frac{n(n+\beta)(\alpha+\beta+2)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta)(\alpha+1)}, \end{aligned}$$

see [9]. The corresponding normalized polynomials are denoted by $R_n^{(\alpha,\beta)}$ and we compute

$$h(0) = 1, \quad h(n) = \frac{(2n+\alpha+\beta+1)\Gamma(\alpha+\beta+n+1)\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{(n+1)\Gamma(\alpha+\beta+2)\Gamma(\alpha+1)\Gamma(\beta+1)}. \quad n \in \mathbb{N},$$

By using Sterling's formula we get

$$h(n) = \mathcal{O}(n^{2\alpha+1}). \tag{34}$$

If $(\alpha, \beta) \in V = \{(\alpha, \beta),\ \alpha \geq \beta > -1,\ \alpha \geq -1/2\}$ then γ is bounded, see [1, Theorem 1], i.e., $(R_n^{(\alpha,\beta)})_{n\in\mathbb{N}_0}$ constitutes a discrete signed hypergroup. Furthermore, if $(\alpha, \beta) \in W = \{(\alpha, \beta),\ \alpha \geq \beta,\ a(a+5)(a+3)^2 \geq (a^2 - 7a - 24)b^2\} \supset \{(\alpha, \beta),\ \alpha \geq \beta > -1,\ \alpha + \beta + 1 \geq 0\}$, where $a=\alpha+\beta+1$ and $b=\alpha-\beta$, then $(R_n^{(\alpha,\beta)})_{n\in\mathbb{N}_0}$ constitutes a discrete hypergroup, see [6, Theorem 1].

By switching the normalization point x_0 to -1 and denoting the corresponding polynomials by $S_n^{(\alpha,\beta)}$ we have $S_n^{(\alpha,\beta)}(x)=R_n^{(\beta,\alpha)}(-x)$, see also [6, p.585]. Hence, when $(\beta,\alpha)\in V$ then $S_n^{(\alpha,\beta)}$ constitute a discrete signed hypergroup and when $(\beta,\alpha)\in W$ they form a discrete hypergroup.

The remaining region is $G = \{(\alpha,\beta),\ -1<\alpha,\beta<-1/2\}$. Making use of Theorem 6.4, (34) and $\gamma(n)\geq \max(h(n)^{-1},1)$ we deduce for $(\alpha,\beta)\in G$ both $\{R_n^{(\alpha,\beta)},\ n\in\mathbb{N}_0\}$ and $\{S_n^{(\alpha,\beta)},\ n\in\mathbb{N}_0\}$ form an SBG polynomial hypergroup which is not bounded.

For the ultraspherical polynomials, i.e., $\alpha=\beta$ we will determine $\gamma(n)$ explicitly for $-1<\alpha<-1/2$.

Theorem 7.1 *Let $-1<\alpha<-1/2$. For γ corresponding with $R_n^{(\alpha,\alpha)}$ it holds $\gamma(0)=\gamma(1)=1$ and for $n\geq 2$*

$$\gamma(n) \;=\; \sum_k |g(n,n,k)| = \frac{2}{\prod_{k=1}^{n-1} a_k}\left(\prod_{k=n}^{2n-1} a_k + \prod_{k=1}^{n} c_k\right) - 1 \;<\; 4h(n)^{-1}-1.$$

In particular there exist constants $C_1,C_2>0$ such that

$$C_1 n^{-2\alpha-1} \;\leq\; \gamma(n) \;\leq\; C_2 n^{-2\alpha-1}. \tag{35}$$

<u>PROOF:</u> Make use of Theorem 6.4, (34) and $\gamma(n)\geq\max(h(n)^{-1},1)$ to show the correspondence with an SBG hypergroup which is not bounded.
It is clear that $\gamma(0)=\gamma(1)=1$. We use Lemma 6.1 to deduce for $n\geq m\geq 2$ that $g(n,m,n-m+2j-1)=0,\ j=1,2,\dots,m,\ g(n,m,n-m),g(n,m,n-m)>0$ and $g(n,m,n-m+2j)<0,\ j=1,2,\dots,m-1$. Hence for all $n,m\geq 2$

$$\sum_k |g(n,m,k)| = 2(g(n,m,|n-m|)+g(n,m,n+m))-1.$$

Let $2\leq m<n$. Using (29) and (30) we derive

$$\sum_k g|(n,m,k)| < \sum_k g|(n,m+1,k)|.$$

Now suppose $2\leq n\leq m$ and set $r_m = \prod_{k=m}^{m+n-1} a_k + \prod_{k=m-n+1}^{m} c_k$. Since $\sum_k |g(n,m,k)| = 2r_m\left(\prod_{k=1}^{n-1} a_k\right)^{-1} - 1$, the inequality $r_{m+1}<r_m$ yields

$$\sum_k |g(n,m+1,k)| \;<\; \sum_k |g(n,m,k)|.$$

Finally, we derive

$$\gamma(n) \;=\; \sum_k |g(n,n,k)| = \frac{2}{\prod_{k=1}^{n-1} a_k}\left(\prod_{k=n}^{2n-1} a_k + \prod_{k=1}^{n} c_k\right) - 1 \quad \text{for all } n\geq 2.$$

By using $\prod_{k=n}^{2n-1} a_k < \prod_{k=1}^{n} c_k$ this yields $h(n)^{-1} \leq \gamma(n) < 4h(n)^{-1} - 1$. With (34) we get the last assertion. ■

Now it is easy to determine the dual objects of the generalized hypergroups generated by ultraspherical polynomials.

Theorem 7.2 *Let $-1 < \alpha$. Then the duals of the generalized hypergroup $\{R_n^{(\alpha,\alpha)},\ n \in \mathbb{N}_0\}$ of ultraspherical polynomials coincide,*

$$\mathcal{S} \;=\; \widehat{\mathbb{N}_0} \;=\; \mathcal{X}^b(\mathbb{N}_0) \;\simeq\; [-1,1]\,.$$

PROOF: We have to show $\mathcal{S} \supset \mathcal{X}^b(K)$. Assume $z \in \mathbb{C} \setminus [-1,1]$. From [16, (8.21.9)] we deduce that $R_n(z)$ grows exponentially with n. From (35) we know that $\gamma(n)$ grows only polynomially. Hence, there does not exist a constant r such that $|R_n(z)| \leq r\gamma(n)$ which means $\alpha_z \notin \mathcal{X}^b(\mathbb{N}_0)$. ■

One might have the question what happens in the case $(\alpha,\beta) \in G$ when choosing the normalization point $c \notin [-1,1]$. Surprisingly, [1, Theorem 2] immediately yields the following theorem.

Theorem 7.3 *Let $(\alpha,\beta) \in G$ and choose $c \in \mathbb{R} \setminus [-1,1]$. The Jacobi polynomials $T_n^{(\alpha,\beta)}$ normalized at c (i.e., $T_n^{(\alpha,\beta)}(c) = 1$) constitute a normalized and bounded generalized hypergroup, i.e., a discrete signed hypergroup. The duals are given by*

$$\mathcal{S} \;=\; [-1,1], \qquad \widehat{\mathbb{N}_0} \;=\; [-|c|,|c|], \qquad \mathcal{X}^b(\mathbb{N}_0) \;=\; \{z \in \mathbb{C},\ |z + \sqrt{z^2-1}| \leq |c| + \sqrt{c^2-1}\}.$$

Hereby, the branch of $\sqrt{z^2-1}$ is chosen such that $|z + \sqrt{z^2-1}| \geq 1$.

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