

Expected dispersion of uniformly distributed points

Aicke Hinrichs*, David Krieg†, Robert J. Kunsch‡, Daniel Rudolf§

November 29, 2019

Abstract

The dispersion of a point set in $[0, 1]^d$ is the volume of the largest axis parallel box inside the unit cube that does not intersect with the point set. We study the expected dispersion with respect to a random set of n points determined by an i.i.d. sequence of uniformly distributed random variables. Depending on the number of points n and the dimension d we provide an upper and lower bound of the expected dispersion. In particular, we show that the minimal number of points required to achieve an expected dispersion less than $\varepsilon \in (0, 1)$ depends linearly on the dimension d .

Keywords: expected dispersion, dispersion, delta cover

Classification. Primary: 62D05; **Secondary:** 52B55, 65Y20, 68Q25.

1 Introduction and main result

For n points $\{x_1, \dots, x_n\} \subset [0, 1]^d$ the dispersion is the volume of the largest axis parallel box that does not contain a point. It is defined by

$$\text{disp}(x_1, \dots, x_n) := \sup_{B \cap \{x_1, \dots, x_n\} = \emptyset} \lambda_d(B), \quad (1)$$

*Institut für Analysis, Johannes Kepler Universität Linz, Altenbergerstrasse 69, 4040 Linz, Austria, Email: aicke.hinrichs@jku.at, A. Hinrichs is supported by the Austrian Science Fund (FWF) Project F5509-N26, which is a part of the Special Research Program “Quasi-Monte Carlo Methods: Theory and Applications”.

†Institut für Analysis, Johannes Kepler Universität Linz, Altenbergerstrasse 69, 4040 Linz, Austria, Email: david.krieg@jku.at, D. Krieg is supported by the Austrian Science Fund (FWF) Project F5509-N26, which is a part of the Special Research Program “Quasi-Monte Carlo Methods: Theory and Applications”.

‡Lehrstuhl für Mathematik der Informationsverarbeitung, Pontdriesch 10, 52062 Aachen, Germany, Email: kunsch@mathc.rwth-aachen.de

§Institute for Mathematical Stochastics, Georg-August-Universität Göttingen, Goldschmidtstraße 7, 37077 Göttingen, Email: daniel.rudolf@uni-goettingen.de

where λ_d denotes the d -dimensional Lebesgue measure and the supremum is taken over all boxes $B = I_1 \times \cdots \times I_d$ with intervals $I_k \subseteq [0, 1]$. In this note we study the expected dispersion of random points based on an i.i.d. sequence of uniformly distributed random variables $(X_i)_{i \in \mathbb{N}}$, where each X_i maps from a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to $[0, 1]^d$. For simplicity we write $X_1, X_2, \dots \sim \text{Unif}([0, 1]^d)$. We ask for the behavior of

$$\mathbb{E}(\text{disp}(X_1, \dots, X_n))$$

in terms of n and d .

In recent years the proof of existence and the construction of point sets with small dispersion attracted considerable attention, see [1, 6, 12, 13, 18, 19]. To describe optimality properties of such point sets of cardinality n in the d -dimensional setting let us define the minimal dispersion

$$\text{disp}(n, d) := \inf_{\{x_1, \dots, x_n\} \subset [0, 1]^d} \text{disp}(x_1, \dots, x_n),$$

and its inverse

$$n(\varepsilon, d) := \min\{n \in \mathbb{N} \mid \text{disp}(n, d) \leq \varepsilon\},$$

where $\varepsilon \in (0, 1)$. A lower bound for the minimal dispersion growing with the dimension d is provided in [1, Theorem 1]. Moreover, [1, Section 4] contains an upper bound due to Gerhard Larcher, based on constructions of digital nets, which give explicitly constructable point sets. For example, for $\varepsilon \in (0, 1/8)$ the bounds are

$$2^{-3}\varepsilon^{-1} \log_2 d \leq n(\varepsilon, d) \leq 2^{7d+1}\varepsilon^{-1}, \quad (2)$$

which shows that the dependence on ε^{-1} cannot be improved. However, the gap w.r.t. the dependence on d motivated the papers [13, 18]. Based on probabilistic arguments, in [13, 18] the existence of “good” point sets is proven. Those results show that for fixed ε the quantity $n(\varepsilon, d)$ increases at most logarithmically in d , so that the d -dependence of the lower bound in (2) cannot be improved. By the use of a derandomization technique, [19] provides a deterministic algorithm for the construction of point sets with cardinality $c_\varepsilon \log_2(d)$ and dispersion at most ε , where $c_\varepsilon > 0$ depends only polynomially on ε . In Table 1 we survey bounds for $n(\varepsilon, d)$, in particular, it contains the ones of [13, 18] and their dependence on ε .

However, all the existence results of “good” point sets rely on randomly drawn points and probabilistic arguments. In particular, the estimate of [12, Corollary 1] is based on an i.i.d. sequence of uniformly distributed random variables. Maybe this is the most canonical randomly chosen point set and one might ask how good it is compared to deterministic point sets. Here the measure of goodness is the expected dispersion and our main result is as follows:

Theorem 1.1. *For any $n > d$ we have*

$$\max \left\{ \frac{\log(n)}{9n}, \frac{d}{2en} \right\} \leq \mathbb{E}(\text{disp}(X_1, \dots, X_n)) \leq \frac{9d}{n} \log \left(\frac{en}{d} \right).$$

Reference	Upper bound of $n(\varepsilon, d)$	Remarks
[1] (due to Larcher)	$2^{7d+1} \varepsilon^{-1}$	optimal in ε^{-1} digital net construction
[6]	$\min\{2 d^{\lceil \log_2(\varepsilon^{-1}) - 1 \rceil}, \varepsilon^{-1} \lceil \log_2(\varepsilon^{-1}) \rceil^{d-1}\}$	sparse grid construction
[12]	$8 d \varepsilon^{-1} \log(33 \varepsilon^{-1})$	existence of point set
[13]	$\log_2(d) \varepsilon^{-(\varepsilon^{-2}+2)} (4 \log \varepsilon^{-1} + 1)$	optimal in d existence of point set
[18]	$2^7 \log_2(d) \varepsilon^{-2} (1 + \log_2(\varepsilon^{-1}))^2$	optimal in d existence of point set

Table 1: The table contains several upper bounds on $n(\varepsilon, d)$ based on existence results of “good” points as well as explicit constructions.

Let us also state our result in terms of the inverse of the expected dispersion. For $\varepsilon \in (0, 1)$ and $d \in \mathbb{N}$, the inverse of the expected dispersion is defined as

$$N(\varepsilon, d) := \min\{n \in \mathbb{N} \mid \mathbb{E} \text{disp}(X_1, \dots, X_n) \leq \varepsilon\}.$$

Corollary 1.2. *For all $\varepsilon \in (0, \frac{1}{9e})$ and $d \in \mathbb{N}$ we have*

$$\max\left\{\frac{1}{9\varepsilon} \log\left(\frac{1}{9\varepsilon}\right), \frac{d}{2e\varepsilon}\right\} \leq N(\varepsilon, d) \leq \left\lceil 9(1 + e^{-1}) \frac{d}{\varepsilon} \log\left(\frac{9(e+1)}{\varepsilon}\right) \right\rceil.$$

These estimates show that $N(\varepsilon, d)$ for fixed ε behaves linearly w.r.t. the dimension, and for fixed d behaves like $\varepsilon^{-1} \log(\varepsilon^{-1})$. It is interesting to note that the linear behavior w.r.t. d is in contrast to the $\log_2(d)$ dependence of the inverse of the minimal dispersion.

The upper bound of Theorem 1.1 follows by exploiting a δ -cover approximation and a concentration inequality stated in [12]. The proof of the lower bound is separated into two parts. First, we derive the bound $\log(n)/(9n)$ from well known results on the coupon collector’s problem. After that the d -dependent lower bound $d/(2en)$ is proven by a reduction to the expected dispersion of d points and, eventually, a constant lower bound for this quantity.

The proof of Theorem 1.1 with the necessary notation is given in Section 2. Further discussions and extensions of the results are provided in Section 3.

2 Proof of Theorem 1.1

2.1 The upper bound

Before we start with the proof of the upper bound let us provide some further notation. Let \mathcal{B} be the set of boxes given as

$$\mathcal{B} := \left\{ \prod_{k=1}^d [a^{(k)}, b^{(k)}) \subseteq [0, 1]^d \mid a^{(k)}, b^{(k)} \in \mathbb{Q} \cap [0, 1], k = 1, \dots, d \right\}.$$

Then, obviously, we have

$$\text{disp}(x_1, \dots, x_n) = \sup_{\substack{B \in \mathcal{B} \\ B \cap \{x_1, \dots, x_n\} = \emptyset}} \lambda_d(B).$$

Note that with this we can restrict ourself to boxes determined by half-open intervals with rational boundary values. Thus, the supremum within the dispersion is only taken over a countable set, which leads to the measurability of the mapping $(x_1, \dots, x_n) \mapsto \text{disp}(x_1, \dots, x_n)$. Occasionally we also call \mathcal{B} the set of test sets. A δ -cover of the set of test sets \mathcal{B} for $\delta \in (0, 1]$ is given by a finite set $\Gamma_\delta \subset \mathcal{B}$ that satisfies

$$\forall B \in \mathcal{B} \quad \exists L_B, U_B \in \Gamma_\delta \quad \text{with} \quad L_B \subseteq B \subseteq U_B \quad \text{and} \quad \lambda(U_B \setminus L_B) \leq \delta.$$

Furthermore, for $x_1, \dots, x_n \in [0, 1]^d$ and a δ -cover Γ_δ for \mathcal{B} with $\delta > 0$ define

$$\text{disp}_\delta(x_1, \dots, x_n) := \sup_{\substack{A \in \Gamma_\delta \\ A \cap \{x_1, \dots, x_n\} = \emptyset}} \lambda_d(A).$$

Having introduced those quantities we state two results from [12]. From Γ_δ being a δ -cover it follows that

$$\text{disp}(x_1, \dots, x_n) \leq \delta + \text{disp}_\delta(x_1, \dots, x_n), \tag{3}$$

and from a union bound, it follows that for any $s \in (0, 1)$ we have

$$\mathbb{P}(\text{disp}_\delta(X_1, \dots, X_n) > s) \leq |\Gamma_\delta| (1 - s)^n. \tag{4}$$

We refer to [12, Lemma 1] and the proof of [12, Theorem 1] for details. These results lead to the following lemma.

Lemma 2.1. *For $\delta > 0$ assume that the set Γ_δ is a δ -cover of \mathcal{B} . Then, for any $n \geq \log |\Gamma_\delta|$ we have*

$$\mathbb{E}(\text{disp}(X_1, \dots, X_n)) \leq \delta + \frac{\log |\Gamma_\delta|}{n} + \frac{1}{n+1}.$$

Proof. From (3) we have

$$\mathbb{E}(\text{disp}(X_1, \dots, X_n)) \leq \delta + \mathbb{E}(\text{disp}_\delta(X_1, \dots, X_n)).$$

Furthermore by using (4) we obtain

$$\begin{aligned} \mathbb{E}(\text{disp}_\delta(X_1, \dots, X_n)) &= \int_0^1 \mathbb{P}(\text{disp}_\delta(X_1, \dots, X_n) > s) \, ds \\ &\leq \frac{\log |\Gamma_\delta|}{n} + \int_{(\log |\Gamma_\delta|)/n}^1 \mathbb{P}(\text{disp}_\delta(X_1, \dots, X_n) > s) \, ds \\ &\leq \frac{\log |\Gamma_\delta|}{n} + |\Gamma_\delta| \int_{(\log |\Gamma_\delta|)/n}^1 (1-s)^n \, ds \leq \frac{\log |\Gamma_\delta|}{n} + \frac{|\Gamma_\delta|}{n+1} \left(1 - \frac{\log |\Gamma_\delta|}{n}\right)^{n+1}. \end{aligned}$$

Note that for any $0 \leq a \leq n$ we have $(1 - a/n)^n \leq \exp(-a)$, such that

$$|\Gamma_\delta| \left(1 - \frac{\log |\Gamma_\delta|}{n}\right)^{n+1} \leq 1,$$

which finishes the proof. \square

Remark 2.2. Except for the assumption that we have a δ -cover we did not use any property of the set of test sets \mathcal{B} .

Now the upper bound of Theorem 1.1 is deduced by the results on δ -covers for \mathcal{B} from Gnewuch, see [4]. Namely, from [4, Formula (1), Theorem 1.15, Lemma 1.18] one obtains that there is a δ -cover for \mathcal{B} with $|\Gamma_\delta| \leq (6e\delta^{-1})^{2d}$. By setting $\delta = 6d/n$, the upper bound of Theorem 1.1 follows with Lemma 2.1 and $n \geq d$.

Finally, this upper estimate implies the upper bound of Corollary 1.2. For the convenience of the reader, we add a few arguments. We are looking for the smallest integer $n \geq d$ such that

$$\frac{9d}{n} \log \left(\frac{en}{d} \right) \leq \varepsilon.$$

Since the left-hand side is monotonically decreasing for $n \geq d$, picking any

$$n \geq c \frac{d}{\varepsilon} \log \left(\frac{ce}{\varepsilon} \right), \quad \text{where } c \geq 1 \quad \text{such that } n \geq d,$$

will lead to

$$\frac{9d}{n} \log \left(\frac{en}{d} \right) \leq \frac{9}{c} \varepsilon \cdot \left(1 + \frac{\log \log \left(\frac{ce}{\varepsilon} \right)}{\log \left(\frac{ce}{\varepsilon} \right)} \right) \leq \frac{9(1 + e^{-1})}{c} \varepsilon,$$

where we used that $(\log x)/x$ attains its maximum for $x = e$. Hence, taking $c = 9(1 + e^{-1}) = 12.31\dots$ we obtain the desired guarantee.

2.2 The lower bound

In Section 2.2.1 we show that $\mathbb{E}(\text{disp}(X_1, \dots, X_n)) \geq \frac{\log(n)}{9n}$, and in Section 2.2.2 we prove that $\mathbb{E}(\text{disp}(X_1, \dots, X_n)) \geq \frac{d}{2en}$ for $n > d$. Both lower bounds together yield the corresponding statement of Theorem 1.1. By convention, all random variables are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

2.2.1 Lower bound without dimension dependence

We start with an auxiliary tool, using results on the *coupon collector's problem*.

Lemma 2.3. *For $\ell \in \mathbb{N}$ let $(Y_i)_{i \in \mathbb{N}}$ be an i.i.d. sequence of uniformly distributed random variables in $\{1, \dots, \ell\}$. Define $H_\ell := \sum_{j=1}^{\ell} j^{-1}$ and*

$$\tau_\ell := \min\{k \in \mathbb{N} \mid \{Y_1, \dots, Y_k\} = \{1, \dots, \ell\}\}.$$

Then, for any integer $n \leq (H_\ell - 2)\ell$ we have $\mathbb{P}(\tau_\ell > n) > 1/2$.

Proof. It is well known that the mean and the variance of τ_ℓ satisfy

$$\mathbb{E} \tau_\ell = \ell H_\ell \quad \text{and} \quad \text{Var} \tau_\ell \leq \ell^2 \sum_{j=1}^{\ell} j^{-2} \leq \frac{\pi^2}{6} \ell^2$$

For details concerning these estimates, see for example [9] or [7, Proposition 4.7]. Then, for $n \leq (H_\ell - 2)\ell$, by Chebyshev's inequality we have

$$\mathbb{P}(\tau_\ell \leq n) \leq \mathbb{P}(\tau_\ell \leq (H_\ell - 2)\ell) = \mathbb{P}(\ell H_\ell - \tau_\ell \geq 2\ell) \leq \frac{\text{Var}(\tau_\ell)}{4\ell^2} \leq \frac{\pi^2}{24} < \frac{1}{2},$$

which finishes the proof. \square

With the previous result we are able to prove the desired lower bound in the following lemma.

Lemma 2.4. *For any integer $n \geq 3$ we have $\mathbb{E}(\text{disp}(X_1, \dots, X_n)) > \frac{\log(n)}{9n}$.*

Proof. For $\ell \in \mathbb{N}$ split $[0, 1]^d$ into ℓ disjoint boxes B_1, \dots, B_ℓ of equal volume $1/\ell$. For $i = 1, \dots, n$ define the random variable $Y_i : \Omega \rightarrow \{1, \dots, \ell\}$ that indicates the box the point X_i lies in, i.e. $X_i(\omega) \in B_{Y_i(\omega)}$. Note that Y_1, \dots, Y_n are i.i.d. and each uniformly distributed in $\{1, \dots, \ell\}$. Furthermore, for $\omega \in \Omega$ satisfying

$$\{Y_1(\omega), \dots, Y_n(\omega)\} \neq \{1, \dots, \ell\},$$

there is an index $r \in \{1, \dots, \ell\}$ such that $\{X_1(\omega), \dots, X_n(\omega)\} \cap B_r = \emptyset$. Thus, for such an ω we obtain

$$\text{disp}(X_1(\omega), \dots, X_n(\omega)) \geq 1/\ell.$$

This yields

$$\begin{aligned}\mathbb{E}(\text{disp}(X_1, \dots, X_n)) &= \int_{\Omega} \text{disp}(X_1(\omega), \dots, X_n(\omega)) \mathbb{P}(d\omega) \\ &\geq \frac{1}{\ell} \mathbb{P}(\{Y_1, \dots, Y_n\} \neq \{1, \dots, \ell\}).\end{aligned}$$

Observe that with τ_ℓ defined in Lemma 2.3 we have

$$\mathbb{P}(\{Y_1, \dots, Y_n\} \neq \{1, \dots, \ell\}) = \mathbb{P}(\tau_\ell > n).$$

Choosing $\ell := \left\lceil \frac{(1+e)n}{\log(n)} \right\rceil$, we get

$$\frac{n}{\ell} \leq \frac{\log(n)}{1+e} \leq \log\left(\frac{(1+e)n}{\log(n)}\right) - 2 \leq \log(\ell) - 2 < H_\ell - 2,$$

where we used the inequality $\log\left(\frac{(1+e)x}{\log(x)}\right) - 2 - \frac{\log(x)}{1+e} \geq 0$ for $x > 1$ (attaining equality in $x = \exp(1 + 1/e)$), as well as $H_\ell = \sum_{j=1}^{\ell} j^{-1} > \log(\ell + 1)$. This asserts $n \leq (H_\ell - 2)\ell$, and by Lemma 2.3 we obtain $\mathbb{P}(\tau_\ell > n) > 1/2$. Taking everything together yields

$$\mathbb{E}(\text{disp}(X_1, \dots, X_n)) > \frac{1}{2\ell} \geq \frac{1}{2} \cdot \frac{\log(n)}{(1+e)n + \log(n)} > \frac{\log(n)}{9n},$$

which completes the proof. Our derivation holds for integers $n \geq 2$, but the bound starts decaying for $n \geq 3$, in the first place. \square

Having the result of the previous lemma, the first part within the maximum of the lower bound in Corollary 1.2 follows. For the convenience of the reader we add a few arguments. If the expected dispersion shall be smaller than a given $\varepsilon > 0$, the number of points, n , must satisfy $\frac{\log n}{9n} \leq \varepsilon$. Note that the left-hand side is monotonically decreasing only for $n \geq e$, but the expected dispersion for $n \in \{1, 2\}$ should be larger or equal the expected dispersion for $n = 3$. Restricting to $\varepsilon \in (0, \frac{1}{9e})$, for $e \leq n < \frac{1}{9\varepsilon} \log\left(\frac{1}{9\varepsilon}\right)$ we would have

$$\frac{\log n}{9n} > \varepsilon \cdot \left(1 + \frac{\log \log\left(\frac{1}{9\varepsilon}\right)}{\log\left(\frac{1}{9\varepsilon}\right)}\right) > \varepsilon.$$

Hence, $n \geq \frac{1}{9\varepsilon} \log\left(\frac{1}{9\varepsilon}\right)$ is necessary for the expected dispersion to be less or equal ε .

2.2.2 Dimension-dependent lower bound

The proof of the lower bound w.r.t. the dimension is separated into two steps. First, we deduce a lower bound of the expected dispersion of n points in terms of the expected dispersion of d points, see Lemma 2.5. Thus, we reduce the problem to finding a lower bound of the expected dispersion of d points, which then is the

goal of the second step, see Lemma 2.6. In the following proof we use for $B \in \mathcal{B}$ and x_1, \dots, x_ℓ with $\ell \in \mathbb{N}$ the notation

$$\text{disp}_{|B}(x_1, \dots, x_\ell) := \sup_{\substack{R \in \mathcal{B} \cap B \\ R \cap \{x_1, \dots, x_\ell\} = \emptyset}} \lambda_d(R)$$

for the dispersion restricted to B . The following reduction lemma is a probabilistic version of [1, Lemma 1].

Lemma 2.5. *For any $n, \ell \in \mathbb{N}$ we have*

$$\mathbb{E}(\text{disp}(X_1, \dots, X_n)) \geq \frac{\ell + 1}{n + \ell + 1} \mathbb{E}(\text{disp}(X_1, \dots, X_\ell)).$$

Proof. We start with a purely combinatorial argument, a version of the pigeonhole principle. If we split $[0, 1]^d$ into m boxes B_1, \dots, B_m of equal volume, then there is some $j \in \{1, \dots, m\}$ such that B_j contains no more than $\lfloor n/m \rfloor$ of the points X_1, \dots, X_n . Choosing $m = \lceil \frac{n+1}{\ell+1} \rceil = \lfloor \frac{n}{\ell+1} \rfloor + 1$, we have $\lfloor \frac{n}{m} \rfloor \leq \lfloor \frac{n}{n+1}(\ell+1) \rfloor \leq \ell$. For $k \in \mathbb{N}$, let $n_k \in \mathbb{N}$ be the time when B_j is hit by the sequence $(X_i)_{i \in \mathbb{N}} \subset [0, 1]^d$ for the k -th time (which for $X_1, X_2, \dots \stackrel{\text{iid}}{\sim} \text{Unif}([0, 1]^d)$ almost surely happens). With $n_\ell \geq n$ (due to the choice of B_j) we obtain

$$\text{disp}(X_1, \dots, X_n) \geq \text{disp}_{|B_j}(\{X_1, \dots, X_n\} \cap B_j) \geq \text{disp}_{|B_j}(X_{n_1}, \dots, X_{n_\ell}).$$

Let T be an affine transformation that maps B_j onto $[0, 1]^d$. Then

$$\text{disp}_{B_j}(X_{n_1}, \dots, X_{n_\ell}) = \lambda_d(B_j) \cdot \text{disp}(TX_{n_1}, \dots, TX_{n_\ell}).$$

Recall that $X_1, X_2, \dots \stackrel{\text{iid}}{\sim} \text{Unif}([0, 1]^d)$, hence the points $TX_{n_1}, \dots, TX_{n_\ell}$ are independent and uniformly distributed in $[0, 1]^d$. Taking the expectation and using $\lambda_d(B_j) = \frac{1}{m} \geq \frac{1}{n/(\ell+1)+1} = \frac{\ell+1}{n+\ell+1}$, yields the statement. \square

Thus, it is sufficient to provide a constant lower bound of the expected dispersion of d points. Slightly more general we obtain the following.

Lemma 2.6. *For any $d, \ell \in \mathbb{N}$ and $X_1, \dots, X_\ell \stackrel{\text{iid}}{\sim} \text{Unif}([0, 1]^d)$ we have*

$$\mathbb{E}(\text{disp}(X_1, \dots, X_\ell)) \geq e^{-\ell/d}.$$

Proof. For all $i \in \{1, \dots, \ell\}$, let X_i^* denote the largest coordinate of X_i , i.e.,

$$X_i^* := \max\{X_i^{(1)}, \dots, X_i^{(d)}\}.$$

We choose $j^*(i) \leq d$ such that $X_i^{(j^*(i))} = X_i^*$. Let us consider the box

$$B = \prod_{j=1}^d [0, a_j),$$

where

$$a_j := \min \{1\} \cup \{X_i^* \mid i \leq \ell \text{ with } j^*(i) = j\}.$$

This box is empty, since for all $i \leq \ell$, we have $X_i^{(j^*(i))} \geq a_{j^*(i)}$ and hence $X_i \notin B$. For an illustration for $d = 2$, see Figure 1. On the other hand, the volume of B is given by

$$\lambda^d(B) = \prod_{j=1}^d a_j = \prod_{i \in I} X_i^*,$$

where I is a suitable subset of $\{1, \dots, \ell\}$. This yields

$$\text{disp}(X_1, \dots, X_\ell) \geq \prod_{i \in I} X_i^* \geq \prod_{i=1}^{\ell} X_i^*.$$

The random numbers X_i^* are independent and beta distributed with parameters $\alpha = d$ and $\beta = 1$, in particular, $\mathbb{E}(X_i^*) = 1 - 1/(d + 1)$. Hence

$$\begin{aligned} \mathbb{E}(\text{disp}(X_1, \dots, X_\ell)) &\geq \prod_{i=1}^{\ell} \mathbb{E}(X_i^*) = \left(1 - \frac{1}{d+1}\right)^\ell = \left(\frac{1}{1+\frac{1}{d}}\right)^\ell \\ &\geq \left(\frac{1}{\exp(1/d)}\right)^\ell = e^{-\ell/d}. \end{aligned} \quad \square$$

The proof of the lower bound follows by setting $\ell = d$ and combining the results of the two lemmas. We readily get

$$\mathbb{E}(\text{disp}(X_1, \dots, X_n)) \geq \frac{d+1}{e(n+d+1)} > \frac{d}{2en},$$

where the last inequality follows from $n > d$. For $\varepsilon \in (0, \frac{1}{2e})$, the respective inverse lower bound $N(\varepsilon, d) \geq \frac{d}{2e\varepsilon}$ is straightforward, where the restriction on ε implies $N(\varepsilon, d) > d$.

3 Notes and remarks

The dispersion of a point set as defined in (1) has been introduced in [11] generalizing the work of [5]. The renewed interest in this quantity emerged from its appearance in the construction of algorithms for the approximation of rank-one tensors, see [2, 8, 10], where the dependence on the dimension is important. It is also related to the universal discretization problem, see [15], and the fixed volume discrepancy, see [14, 16].

The dispersion of a point set has also been studied on the torus instead of the unit cube, see for example [3, 17]. This can be translated to the unit cube with another set of test sets given by

$$\tilde{\mathcal{B}} := \left\{ \prod_{k=1}^d I_k(x, y) \mid x = (x^{(1)}, \dots, x^{(d)}), y = (y^{(1)}, \dots, y^{(d)}) \in [0, 1]^d \cap \mathbb{Q}^d \right\},$$

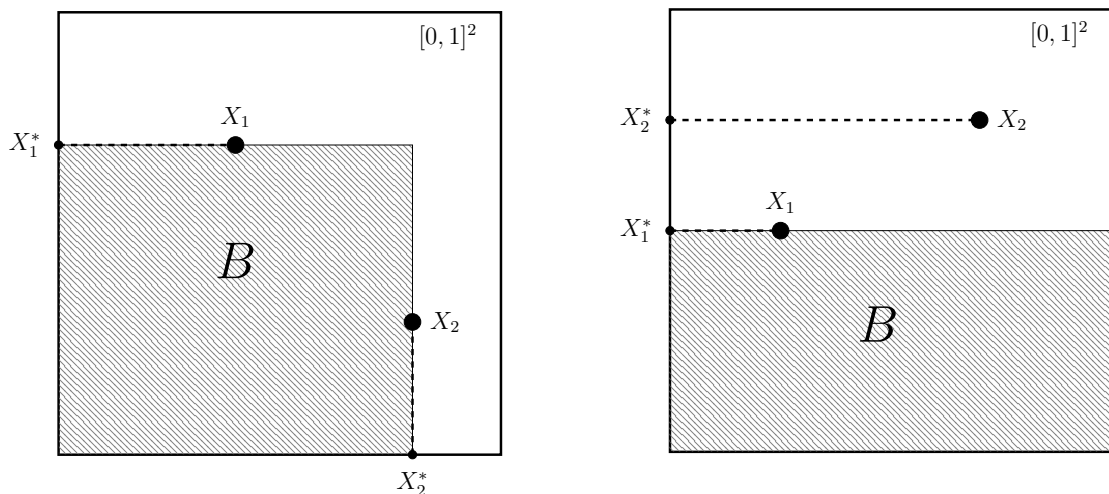


Figure 1: An illustration of the empty box construction from Lemma 2.6 in two situations for $d = \ell = 2$. In the left picture we have $B = [0, a_1) \times [0, a_2)$ with $X_1 = (0.4, 0.7)$, $j^*(1) = 2$, $a_1 = 0.7$ and $X_2 = (0.8, 0.3)$, $j^*(2) = 1$, $a_2 = 0.8$. In the right picture we have $B = [0, a_1) \times [0, a_2)$ with $X_1 = (0.25, 0.5)$, $j^*(1) = 2$, $a_1 = 0.5$ and $X_2 = (0.7, 0.75)$, $j^*(1) = 2$, $a_2 = 1$.

with

$$I_k(x, y) = \begin{cases} (x^{(k)}, y^{(k)}) & x^{(k)} < y^{(k)} \\ [0, 1] \setminus [y^{(k)}, x^{(k)}] & y^{(k)} \leq x^{(k)}. \end{cases}$$

The set $\tilde{\mathcal{B}}$ is called the test set of periodic boxes. Since the proof of the upper bound of the expected dispersion depends on \mathcal{B} only through the δ -cover, with the same arguments we can also derive an upper bound of the expected dispersion w.r.t. $\tilde{\mathcal{B}}$ by using an appropriate periodic δ -cover. For $x_1, \dots, x_n \in [0, 1]^d$ define

$$\widetilde{\text{disp}}(x_1, \dots, x_n) := \sup_{\substack{B \in \tilde{\mathcal{B}} \\ B \cap \{x_1, \dots, x_n\} = \emptyset}} \lambda_d(B).$$

With [12, Lemma 2] we obtain that there is a δ -cover of at most cardinality $(4d\delta^{-1})^{2d}$ of $\tilde{\mathcal{B}}$, such that with $\delta = 2d/n$ we have

$$\mathbb{E}(\widetilde{\text{disp}}(X_1, \dots, X_n)) \leq \frac{5d}{n} \log(2n).$$

By the fact that $\mathcal{B} \subset \tilde{\mathcal{B}}$ we obtain for any $x_1, \dots, x_n \in [0, 1]^d$ that

$$\text{disp}(x_1, \dots, x_n) \leq \widetilde{\text{disp}}(x_1, \dots, x_n),$$

such that the lower bounds of Theorem 1.1 also carry over to $\mathbb{E}(\widetilde{\text{disp}}(X_1, \dots, X_n))$. Here it is worth mentioning that the lower bound w.r.t. the dimension can also

be deduced from [17, Theorem 1]. Thus, in this setting also a linear dimension-dependence is present in $\mathbb{E}(\widetilde{\text{disp}}(X_1, \dots, X_n))$. However, concerning the inverse of the expected dispersion in the periodic case, the precise growth w.r.t. the dimension remains open, we only know that it is between d and $d \log(d)$.

References

- [1] Ch. Aistleitner, A. Hinrichs, and D. Rudolf, *On the size of the largest empty box amidst a point set*, Discrete Appl. Math. **230** (2017), 146–150.
- [2] M. Bachmayr, W. Dahmen, R. DeVore, and L. Grasedyck, *Approximation of high-dimensional rank one tensors*, Constructive Approximation **39** (2014), no. 2, 385–395.
- [3] S. Breneis and A. Hinrichs, *Fibonacci lattices have minimal dispersion on the two-dimensional torus*, arXiv preprint arXiv:1905.03856 (2019).
- [4] M. Gnewuch, *Bracketing numbers for axis-parallel boxes and applications to geometric discrepancy*, J. Complexity **24** (2008), 154–172.
- [5] E. Hlawka, *Abschätzung von trigonometrischen Summen mittels diophantischer Approximationen*, Österreich. Akad. Wiss. Math.-Naturwiss. Kl. S.-B. II, **185** (1976), 43–50.
- [6] D. Krieg, *On the dispersion of sparse grids*, J. Complexity **45** (2018), 115–119.
- [7] D. Krieg, *Algorithms and Complexity for some Multivariate Problems*, arXiv preprint arXiv:1905.01166 (2019).
- [8] D. Krieg and D. Rudolf, *Recovery algorithms for high-dimensional rank one tensors*, J. Approx. Theory **237** (2019), 17–29.
- [9] D. Levin, Y. Peres, and E. Wilmer, *Markov chains and mixing times*, American Mathematical Society, Providence, RI, 2009.
- [10] E. Novak and D. Rudolf, *Tractability of the approximation of high-dimensional rank one tensors*, Constructive Approximation **43** (2016), no. 1, 1–13.
- [11] G. Rote and R. Tichy, *Quasi-monte carlo methods and the dispersion of point sequences*, Math. Comput. **23** (1996), no. 8-9, 9–23.
- [12] D. Rudolf, *An upper bound of the minimal dispersion via delta covers*, Contemporary computational mathematics—a celebration of the 80th birthday of Ian Sloan. Vol. 1, 2, Springer, Cham, 2018, pp. 1099–1108.
- [13] J. Sosnovec, *A note on the minimal dispersion of point sets in the unit cube*, Eur. J. Combin. **69** (2018), 255–259.

- [14] V.N. Temlyakov, *Fixed volume discrepancy in the periodic case*, arXiv preprint arXiv:1710.11499 (2017).
- [15] V.N. Temlyakov, *Universal discretization*, J. Complexity **47** (2018), 97–109.
- [16] V.N. Temlyakov and M. Ullrich, *On the fixed volume discrepancy of the fibonacci sets in the integral norms*, arXiv preprint arXiv:1908.04658 (2019).
- [17] M. Ullrich, *A lower bound for the dispersion on the torus*, Mathematics and Computers in Simulation, in press (2015).
- [18] M. Ullrich and J. Vybiral, *An upper bound on the minimal dispersion*, J. Complexity **45** (2018), 120–126.
- [19] ———, *Deterministic constructions of high-dimensional sets with small dispersion*, Preprint, Available at <https://arxiv.org/abs/1901.06702> (2019).